

Algorithm NCL for constrained optimization

Michael Saunders

MS&E and ICME, Stanford University

Ding Ma, Ken Judd, and Dominique Orban

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Abstract

Standard optimization solvers have difficulty if the active-constraint gradients are not independent at a solution. For example, problems of the form

$$\min f(x) \quad \text{st} \quad c(x) \geq 0 \quad (m \text{ constraints and } n \text{ variables})$$

may have more than n constraints active at a solution. Such problems arise in the modeling of tax policy (with perhaps millions of constraints and thousands of variables).

Algorithm NCL solves a sequence of about 10 augmented Lagrangian subproblems with constraints $c(x) + r \geq 0$. The extra variables r make the constraints linearly independent, and the subproblem solutions converge to the required solution as r is driven to zero. Assuming second derivatives are available, NCL expands the use of interior methods for large-scale optimization.

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LANCELOT's BCL algorithm for general NLP

Conn, Gould & Toint (1992)

LANCELOT

$$\min \phi(x) \quad \text{st} \quad c(x) = 0, \quad \ell \leq x \leq u$$

BCL subproblems (Bound-Constrained augmented Lagrangian):

$$\begin{array}{ll} \text{BC}_k & \underset{x}{\text{minimize}} \quad \phi(x) - y_k^T c(x) + \frac{1}{2} \rho_k \|c(x)\|^2 \\ & \text{subject to} \quad \ell \leq x \leq u \end{array}$$

Loop:

solve BC_k to get x_k^*	decreasing opttol ω_k
if $\ c(x_k^*)\ \leq \eta_k$, $y_{k+1} \leftarrow y_k - \rho_k c(x_k^*)$	decreasing featol η_k
else $\rho_{k+1} \leftarrow 10\rho_k$	

Our optimization problem

Our NLP problem

NLP	$\begin{array}{ll}\text{minimize} & \phi(x) \\ & x \in \mathbb{R}^n \\ \text{subject to} & c(x) \geq 0, \quad \ell \leq x \leq u\end{array}$
-----	--

- $\phi(x)$ is a smooth nonlinear function
- $c(x) \in \mathbb{R}^m$ is a vector of smooth nonlinear functions
- General bounds
- Many inequalities $c(x) \geq 0$ might not satisfy LICQ at x^*

Example: $m = 571,000$, $n = 1500$
10,000 constraints essentially active: $c_i(x^*) \leq 10^{-6}$

BCL

LCL

NCL

Sequence of subproblems minimizing
X-constrained (augmented) Lagrangian

BCL LANCELOT Conn, Gould & Toint (1992)

LCL linearized constraints Robinson (1972)
MINOS Murtagh and S (1982)

sLCL KNOSSOS Friedlander (2002)

NCL New form of **BCL!** Today's talk
AMPL loop + IPOPT or KNITRO

Algorithm NCL for general NLP

NCL subproblems

NLP

minimize $\phi(x)$

subject to $c(x) = 0, \quad \ell \leq x \leq u$

NCL subproblems

NLP

$$\begin{array}{ll}\underset{x}{\text{minimize}} & \phi(x) \\ \text{subject to} & c(x) = 0, \quad \ell \leq x \leq u\end{array}$$

LANCELOT-type subproblems:

BC_k

$$\begin{array}{ll}\underset{x}{\text{minimize}} & L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \frac{1}{2} \rho_k \|c(x)\|^2 \\ \text{subject to} & \ell \leq x \leq u\end{array}$$

NCL subproblems

NLP

$$\begin{aligned} & \underset{x}{\text{minimize}} && \phi(x) \\ & \text{subject to} && c(x) = 0, \quad \ell \leq x \leq u \end{aligned}$$

LANCELOT-type subproblems:

BC_k

$$\begin{aligned} & \underset{x}{\text{minimize}} && L(x, y_k, \rho_k) = \phi(x) - y_k^T c(x) + \frac{1}{2} \rho_k \|c(x)\|^2 \\ & \text{subject to} && \ell \leq x \leq u \end{aligned}$$

Introduce $r = -c(x)$:

NC_k

$$\begin{aligned} & \underset{x, r}{\text{minimize}} && \phi(x) + y_k^T r + \frac{1}{2} \rho_k \|r\|^2 \\ & \text{subject to} && c(x) + r = 0, \quad \ell \leq x \leq u \end{aligned}$$

Free vars r make the nonlinear constraints independent and feasible

Solvers happy!

NCL subproblems

NLP

$$\begin{array}{ll}\text{minimize} & \phi(x) \\ \text{subject to} & c(x) = 0, \quad \ell \leq x \leq u\end{array}$$

NC_k

$$\begin{array}{ll}\text{minimize} & \phi(x) + y_k^T r + \frac{1}{2} \rho_k \|r\|^2 \\ \text{subject to} & c(x) + r = 0, \quad \ell \leq x \leq u\end{array}$$

Free vars r make the nonlinear constraints independent and feasible

Solvers happy!

NCL subproblems for our problem

NLP

$$\begin{array}{ll}\text{minimize} & \phi(x) \\ \text{subject to} & c(x) \geq 0, \quad \ell \leq x \leq u\end{array}$$

NC_k

$$\begin{array}{ll}\text{minimize} & \phi(x) + y_k^T r + \frac{1}{2} \rho_k \|r\|^2 \\ \text{subject to} & c(x) + r \geq 0, \quad \ell \leq x \leq u\end{array}$$

Free vars r make the nonlinear constraints independent and feasible

Solvers happy!

Subproblems for convex QP

QP

minimize $\phi(x)$ subject to $Ax = b, \quad \ell \leq x \leq u$

Chris Maes, ICME PhD thesis (2010)

QPBLUR

QP_kminimize $\phi(x) + y_k^T r + \frac{1}{2} \rho_k \|r\|^2$ subject to $Ax + r = b, \quad \ell \leq x \leq u$ Free vars r make the constraints independent and feasible

Solvers happy!

Optimal Tax Policy

Kenneth Judd and Che-Lin Su 2011



Optimal tax policy

TAX	$\begin{aligned} &\text{maximize}_{c, y} && \sum_i \lambda_i U^i(c_i, y_i) \\ &\text{subject to} && U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0 \quad \text{for all } i, j \text{ (*)} \\ &&& \lambda^T(y - c) \geq 0 \\ &&& c, y \geq 0 \end{aligned}$
-----	--

where c_i and y_i are the consumption and income of taxpayer i , and λ is a vector of positive weights. The utility functions $U^i(c_i, y_i)$ are each of the form

$$U(c, y) = \frac{(c - \alpha)^{1-1/\gamma}}{1 - 1/\gamma} - \psi \frac{(y/w)^{1/\eta+1}}{1/\eta + 1}$$

where w is the wage rate and α , γ , ψ and η are taxpayer heterogeneities

(*) = incentive-compatibility or self-selection constraints (zillions of them)

Optimal tax policy

More precisely,

$$U^{i,j,k,g,h}(c_{p,q,r,s,t}, y_{p,q,r,s,t}) = \frac{(c_{p,q,r,s,t} - \alpha_k)^{1-1/\gamma_h}}{1 - 1/\gamma_h} - \psi_g \frac{(y_{p,q,r,s,t}/w_i)^{1/\eta_j+1}}{1/\eta_j + 1}$$

where (i, j, k, g, h) and (p, q, r, s, t) run over 5 dimensions:

<i>na</i>	wage types	= 5	21
<i>nb</i>	elasticities of labor supply	= 3	3
<i>nc</i>	basic need types	= 3	3
<i>nd</i>	levels of distaste for work	= 2	2
<i>ne</i>	elasticities of demand for consumption	= 2	2
<i>T</i> =	<i>na</i> × <i>nb</i> × <i>nc</i> × <i>nd</i> × <i>ne</i>	= 180	756
<i>m</i> =	<i>T</i> (<i>T</i> − 1) nonlinear constraints	= 32220	570780
<i>n</i> =	2 <i>T</i> variables	= 360	1512

AMPL model

TAX	$\begin{aligned} &\text{maximize}_{c,y} && \sum_i \lambda_i U^i(c_i, y_i) \\ &\text{subject to} && U^i(c_i, y_i) - U^i(c_j, y_j) \geq 0 \quad \text{for all } i, j \\ & && \lambda^T (y - c) \geq 0 \\ & && c, y \geq 0 \end{aligned}$
-----	--

Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:

!(i=p and j=q and k=r and g=s and h=t)}:

(c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])

- psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]

- (c[p,q,r,s,t] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])

+ psi[g]*(y[p,q,r,s,t]/w[i])^mu1[j] / mu1[j]

>= 0;

Technology:

sum{(i,j,k,g,h) in T} lambda[i,j,k,g,h]*(y[i,j,k,g,h] - c[i,j,k,g,h]) >= 0;

Piecewise-smooth extension

```

Incentive{(i,j,k,g,h) in T, (p,q,r,s,t) in T:
    !(i=p and j=q and k=r and g=s and h=t)}:
    (if c[i,j,k,g,h] - alpha[k] >= epsilon then
        (c[i,j,k,g,h] - alpha[k])^(1-1/gamma[h]) / (1-1/gamma[h])
        - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    else
        - 0.5/gamma[h] *epsilon^(-1/gamma[h]-1)*(c[i,j,k,g,h] - alpha[k])^2
        + (1+1/gamma[h])*epsilon^(-1/gamma[h])*(c[i,j,k,g,h] - alpha[k])
        + (1/(1-1/gamma[h]) - 1 - 0.5/gamma[h])*epsilon^(1-1/gamma[h])
        - psi[g]*(y[i,j,k,g,h]/w[i])^mu1[j] / mu1[j]
    )
- (if c[p,q,r,s,t] - alpha[k] >= epsilon then
    ...
) >= 0;

```

SNOPT on problem TAX (1st derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2$ $m = 32220$ $n = 360$

Major	Minors	Step	nCon	Feasible	Optimal	MeritFunction	nS	condHz	Penalty			
0	866		1	(3.7E-15)	4.9E-04	4.1745522E+02	4	4.1E+08	1.0E+04	_	r	t
1	503	2.7E-02	6	(3.6E-15)	6.5E-02	4.1746922E+02	24	3.2E+05	1.0E+04	_n	r	t
2	134	1.0E-01	11	(1.4E-07)	2.7E-05	4.1755749E+02	8	2.6E+09	1.8E+06	_s		
3	313	9.8E-02	16	(1.4E-07)	8.9E-05	4.1764438E+02	43	1.0E+07	1.8E+06	_		
4	153	2.8E-02	21	(5.5E-08)	1.8E-04	4.1767129E+02	35	2.2E+04	1.8E+06	_		
5	103	2.2E-02	26	(5.4E-08)	9.5E-04	4.1769616E+02	34	6.7E+07	1.8E+06	_		
194	30811	1.0E+00	795	8.6E-01	9.7E-01	2.8330244E+21	2	1.8E+01	3.5E+13	_n		it
195	1819	1.1E-04	800	8.6E-01	1.0E+00	2.6326936E+22	3	1.4E+02	1.1E+15	_n	R	it
195	3314		800	8.6E-01	1.0E+00	2.8661156E+22			1.0E+04	_n	r	it
195	4439		800	8.6E-01	9.9E-01	2.8661156E+22			1.0E+04	_n	r	it

SNOPTB EXIT 40 -- terminated after numerical difficulties

SNOPTB INFO 41 -- current point cannot be improved

IPOPT on problem TAX (2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2$ $m = 32220$ $n = 360$

This is Ipopt version 3.12.4, running with linear solver mumps.

iter	objective	inf_pr	inf_du	lg(mu)	d	lg(rg)	alpha_du	alpha_pr	ls
0	-4.1745522e+02	0.00e+00	2.52e+00	-1.0	0.00e+00	-	0.00e+00	0.00e+00	0
1	-4.1734473e+02	6.18e-03	7.36e+00	-1.0	1.34e+00	-	7.69e-01	2.05e-01f	1
2	-4.1682694e+02	4.93e-03	1.78e+01	-1.0	5.48e+00	-	2.23e-01	1.34e-01f	1
10	-4.1428766e+02	1.22e-03	1.50e+04	-1.0	3.01e-01	0.6	4.75e-01	5.39e-01h	1
160	-4.1641067e+02	0.00e+00	1.50e-03	-3.8	1.25e-01	-	1.00e+00	1.00e+00f	1
449r	-4.1630403e+02	1.13e-05	2.79e-05	-8.1	2.92e-01	-	1.00e+00	9.77e-01h	1

	(scaled)	(unscaled)
Dual infeasibility.....:	1.1130803588695777e+00	1.1130803588695777e+00
Constraint violation.....:	0.0000000000000000e+00	0.0000000000000000e+00
Complementarity.....:	1.3412941119075164e-08	1.3412941119075164e-08

LANCELOT on problem TAX (2nd derivs)

$na, nb, nc, nd, ne = 5, 3, 3, 2, 2$ $m = 32220$ $n = 360$

k	rhok	omegak	etak	Obj	itns	CGit	TRradius	active
1	1.0e+1	1.0e-1	1.0e-1	-417.455	18	12000	4.1e-01	2831
2	1.0e+1	1.0e-2	1.2e-2	-421.606	39	9000	1.6e-01	2568
3	1.0e+2	1.0e-2	7.9e-2	-421.011	23	11000	2.4e-01	1662
4	1.0e+2	1.0e-4	1.3e-3	-420.188	282	104000	8.6e-02	1444
5	1.0e+3	1.0e-3	6.3e-2	-419.967	134	64000	5.7e-02	1004
6	1.0e+3	1.0e-6	1.3e-4	-419.819	198	156000	3.1e-02	901
7	1.0e+4	1.0e-4	5.0e-2	-419.741	300	308000	3.1e-12	710
8	1.0e+4	1.0e-6	1.3e-5	-419.698	327	623000	5.5e-04	709
9	1.0e+5	1.0e-5	4.0e-2	-419.682	253	724000	4.7e-03	653
10	1.0e+5	1.0e-6	1.3e-6	-419.676	154	1031000	4.2e-11	663
11	1.0e+6	1.0e-6	3.2e-2	...				

1970 iterations, 8 hours CPU on NEOS

AMPL implementation of NCL

pTax5Dnclipopt.run

```
reset;  model pTax5Dinitial.run;

reset;  model pTax5Dncl.mod;
        data pTax5Dncl.dat;
        data; var include p5Dinitial.dat;

model;  option solver ipopt;
        option show_stats 1;
        option ipopt_options 'dual_inf_tol=1e-6    max_iter=5000';
```

pTax5Dnclipopt.run

```
option opt2 $ipopt_options ' warm_start_init_point=yes';

for {K in 1..kmax}
{  if K == 2 then {option ipopt_options $opt2 ' mu_init=1e-4'};
   if K == 4 then {option ipopt_options $opt2 ' mu_init=1e-5'};
   if K == 6 then {option ipopt_options $opt2 ' mu_init=1e-6'};
   if K == 8 then {option ipopt_options $opt2 ' mu_init=1e-7'};
   if K ==10 then {option ipopt_options $opt2 ' mu_init=1e-8'};

   solve;

   let rmax := max({(i,j,k,g,h) in T, (p,q,r,s,t) in T:
      !(i=p and j=q and k=r and g=s and h=t)} R[i,j,k,g,h,p,q,r,s,t]);
   let rmin := ...
   let rnorm := max(abs(rmax), abs(rmin));    # ||r||_inf
   if rnorm <= rtol then { printf "Stopping: rnorm is small\n"; break; }
```

pTax5Dnclipopt.run

if $\|r_k^*\| \leq \eta_k$, $y_{k+1} \leftarrow y_k + \rho_k r_k^*$

```
if rnorm <= etak then    # update dual estimate dk; save new solution
{let {(i,j,k,g,h) in T, (p,q,r,s,t) in T:
    !(i=p and j=q and k=r and g=s and h=t)}
    dk[i,j,k,g,h,p,q,r,s,t] :=
    dk[i,j,k,g,h,p,q,r,s,t] + rhok*R[i,j,k,g,h,p,q,r,s,t];
let {(i,j,k,g,h) in T} ck[i,j,k,g,h] := c[i,j,k,g,h];
let {(i,j,k,g,h) in T} yk[i,j,k,g,h] := y[i,j,k,g,h];

if etak == etamin then { printf "Stopping: etak = etamin\n"; break; }
let etak := max(etak*etafac, etamin);
}
```

pTax5Dnclipopt.run

```
else       $\rho_{k+1} \leftarrow 10\rho_k$ 

else      # keep previous solution; increase rhok
{ let {(i,j,k,g,h) in T} c[i,j,k,g,h] := ck[i,j,k,g,h];
  let {(i,j,k,g,h) in T} y[i,j,k,g,h] := yk[i,j,k,g,h];

  if rhok == rhomax then { printf "Stopping: rhok = rhomax\n"; break; }
  let rhok := min(rhok*rhofac, rhomax);
}
} # end main loop
```

Numerical results

Interior Methods (IPMs)

FOLKLORE: We don't know how to warm-start IPMs

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NCL:

- Sequence of related problems

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- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

Interior Methods (IPMs)

FOLKLORE: We don't know how to warm-start IPMs

NCL:

- Sequence of related problems
- Only the objective changes
- Many extra variables r
- r stabilizes iterations, doesn't affect sparsity of factorizations

Maybe warm starts are practical after all!

Warm-start options for Nonlinear Interior Methods

```
IPOPT      warm_start_init_point=yes  
            mu_init=1e-4                (1e-5, ..., 1e-8)
```

NCL/IPOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

k	ρ_k	η_k	$\ r_k^*\ _\infty$	$\phi(x_k^*)$	μ_{init}	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	41.1
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.2
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.1
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-4}	48	25.0
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-4}	43	20.5
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-4}	64	32.9
7	10^5	10^{-5}	6.6e-05	-4.1967177e+02	10^{-4}	57	26.8
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-4}	87	46.2
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-4}	96	53.6

527 iterations, 5 mins CPU

NCL/IPOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

k	ρ_k	η_k	$\ r_k^*\ _\infty$	$\phi(x_k^*)$	μ_{init}	Itns	Time
1	10^2	10^{-2}	7.0e-03	-4.2038075e+02	10^{-1}	95	40.8
2	10^2	10^{-3}	4.1e-03	-4.2002898e+02	10^{-4}	17	7.0
3	10^3	10^{-3}	1.3e-03	-4.1986069e+02	10^{-4}	20	8.5
4	10^4	10^{-3}	4.4e-04	-4.1972958e+02	10^{-5}	57	32.6
5	10^4	10^{-4}	2.2e-04	-4.1968646e+02	10^{-5}	29	14.6
6	10^5	10^{-4}	9.8e-05	-4.1967560e+02	10^{-6}	36	18.7
7	10^5	10^{-5}	3.9e-05	-4.1967205e+02	10^{-6}	35	19.7
8	10^6	10^{-5}	4.2e-06	-4.1967150e+02	10^{-7}	18	7.7
9	10^6	10^{-6}	9.4e-07	-4.1967138e+02	10^{-7}	15	6.8

322 iterations, 3 mins CPU

NCL/IPOPT on problem TAX

$$na, nb, nc, nd, ne = 5, 3, 3, 2, 2 \quad m = 32220 \quad n = 360$$

Constraints within tol of being active: $c_i(x) \leq tol$

<i>tol</i>	<i>count</i>	<i>count/n</i>
10^{-10}	548	1.5
10^{-9}	550	1.5
10^{-8}	591	1.6
10^{-7}	890	2.5
→ 10^{-6}	1104	3.1 ←
10^{-5}	1225	3.4
10^{-4}	1301	3.6
10^{-3}	1655	4.6
10^{-2}	3483	9.7
10^{-1}	10280	28.6

About $3n$ active constraints

NCL/IPOPT bigger example

$na, nb, nc, nd, ne = 21, 3, 3, 2, 2$ $m = 570780$ $n = 1512$

k	ρ_k	η_k	$\ r_k^*\ _\infty$	$\phi(x_k^*)$	μ_init	Itns	Time
1	10^2	10^{-2}	5.1e-03	-1.7656816e+03	10^{-1}	825	7763
2	10^2	10^{-3}	2.4e-03	-1.7648480e+03	10^{-4}	66	473
3	10^3	10^{-3}	1.3e-03	-1.7644006e+03	10^{-4}	106	771
4	10^4	10^{-3}	3.8e-04	-1.7639491e+03	10^{-5}	132	1347
5	10^4	10^{-4}	3.2e-04	-1.7637742e+03	10^{-5}	229	2451
6	10^5	10^{-4}	8.6e-05	-1.7636804e+03	10^{-6}	104	1097
7	10^5	10^{-5}	4.9e-05	-1.7636469e+03	10^{-6}	143	1633
8	10^6	10^{-5}	1.5e-05	-1.7636252e+03	10^{-7}	71	786
9	10^7	10^{-5}	2.8e-06	-1.7636196e+03	10^{-7}	67	726
10	10^7	10^{-6}	5.1e-07	-1.7636187e+03	10^{-8}	18	171

1761 iterations, 5 hours CPU

NCL/IPOPT bigger example

$$na, nb, nc, nd, ne = 21, 3, 3, 2, 2 \quad m = 570780 \quad n = 1512$$

Constraints within tol of being active: $c_i(x) \leq tol$

<i>tol</i>	<i>count</i>	<i>count/n</i>	
10^{-10}	3888	2.6	
10^{-9}	3941	2.6	
10^{-8}	4430	2.9	
10^{-7}	7158	4.7	
→ 10^{-6}	10074	6.6	← $\approx 6.6n$ active constraints
10^{-5}	11451	7.6	
10^{-4}	13109	8.7	
10^{-3}	23099	15.3	
10^{-2}	66361	43.9	
10^{-1}	202664	134.0	

Warm-start options for Nonlinear Interior Methods

IPOPT warm_start_init_point=yes
 mu_init=1e-4 (1e-5, ..., 1e-8)

KNITRO algorithm=1 Thanks, Richard Waltz!
 bar_directinterval=0
 bar_initpt=2
 bar_murule=1
 bar_initmu=1e-4 (1e-5, ..., 1e-8)
 bar_slackboundpush=1e-4 (1e-5, ..., 1e-8)

Comparison of IPOPT, KNITRO, NCL (2nd derivs)

$na = \text{increasing}$			$nb = 3$		$nc = 3$		$nd = 2$		$ne = 2$	
			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	m	n	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	2320	8.0mins
9	104652	648	> 98*	> 360*	928	825	655	1023	9697	1.9hrs
11	156420	792	> 87*	$\infty!$	2769	4117	727	1679	26397	7.0hrs
17	373933	1224			2598	11447	1021	6347		
21	570780	1512					1761	17218	45039	1.9 days

*duals diverge

MUMPS needs more mem

!Loop

Warm starts

Cold starts

NCL/KNITRO with Warm Starts

$na = \text{increasing}$			$nb = 3$		$nc = 3$		$nd = 2$		$ne = 2$	
			IPOPT		KNITRO		NCL/IPOPT		NCL/KNITRO	
na	m	n	itns	time	itns	time	itns	time	itns	time
5	32220	360	449	217	168	53	322	146	339	63
9	104652	648	> 98*	> 360*	928	825	655	1023	307	239
11	156420	792	> 87*	$\infty!$	2769	4117	727	1679	383	420
17	373933	1224			2598	11447	1021	6347	486	1200
21	570780	1512					1761	17218	712	2880

Warm starts

Warm starts

Related work

- C. M. Maes, A Regularized Active-Set Method for Sparse Convex Quadratic Programming. PhD thesis, ICME, Stanford University, 2010.
- M. P. Friedlander and D. Orban, A primal-dual regularized interior-point method for convex quadratic programs. Math. Prog. Comp., 4(1):71–107, 2012.
- S. Arreckx and D. Orban, A regularized factorization-free method for equality-constrained optimization, Technical Report GERAD G-2016-65, GERAD, Montréal, QC, Canada, 2016, doi:10.13140/RG.2.2.20368.00007.
- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: global convergence, IMA J. Numer. Anal., 37 (2017), 407–443.
- P. E. Gill, V. Kungurtsev, and D. P. Robinson, A stabilized SQP method: superlinear convergence, Math. Program., Ser. A, 163 (2017), 369–410.
- O. Hinder and Y. Ye, A one-phase IPM for nonconvex optimization, Oliver's MS&E PhD thesis (2018).

Future possibilities

Solving NCL subproblems with SNOPT

$$\begin{array}{ll} \text{NC}_k & \underset{x, r}{\text{minimize}} \quad \phi(x) + y_k^T r + \frac{1}{2} \rho_k \|r\|^2 \\ & \text{subject to} \quad c(x) + r \geq 0, \quad \ell \leq x \leq u \end{array}$$

Active-set SQP solver should be able to warm-start each NC_k

- **SNOPT7 with SQOPT:**

Many superbasic variables $\Rightarrow$ large reduced Hessian
Quasi-Newton $\Rightarrow$ many minor iterations

- **SNOPT9 with SQIC (Gill and Wong 2014):**

Factorizes QP KKT system $\Rightarrow$ many superbasics OK
2nd derivatives should reduce iterations

- Will need AMPL/SNOPT9 interface (Fortran 2003)

KNOSSOS via AMPL + IPM

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

$$\begin{array}{ll} \text{ELC}_k & \underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) + \sigma_k \|\bar{c}_k(x)\|_1 \\ & \text{subject to} \quad \ell \leq x \leq u \end{array}$$

- $\bar{c}_k(x)$ = linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

KNOSSOS via AMPL + IPM

Stablized LCL (Friedlander and S, 2005) is equivalent to a BCL method:

$$\begin{array}{ll} \text{ELC}_k & \underset{x}{\text{minimize}} \quad L(x, y_k, \rho_k) + \sigma_k \|\bar{c}_k(x)\|_1 \\ & \text{subject to} \quad \ell \leq x \leq u \end{array}$$

- $\bar{c}_k(x)$ = linear approximation to $c(x)$ at x_k (MINOS has very big σ_k)

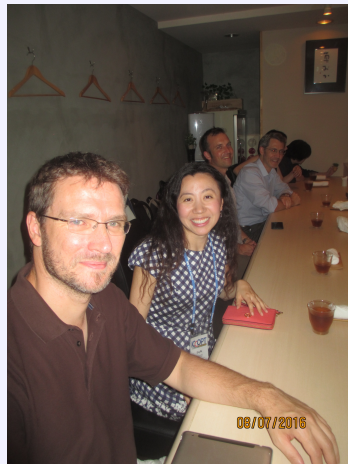
$$\begin{array}{ll} \text{ELC}_k'' & \underset{x, v, w}{\text{minimize}} \quad M_k(x, v, w) + \sigma_k e^T(v + w) \\ & \text{subject to} \quad \bar{c}_k(x) + v - w = 0, \quad \ell \leq x \leq u, \quad v, w \geq 0 \end{array}$$

- M_k = modified augmented Lagrangian $d_k(x, v, w) = c(x) - \bar{c}_k(x) - v + w$
- Subproblem constraints are linear
- IPOPT, KNITRO, ... won't mind the extra elastic variables v, w

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- LANCELOT: Andy Conn, Nick Gould, Philippe Toint
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- Coauthors: Ken Judd, Che-Lin Su, Ding Ma, Dominique Orban
- Yuja Wang, YouTube

Coauthors Ken, Che-Lin, Dominique, Ding



YouTube companion Yuja Wang

