Global Minimum Peak RF Design for Large Time-Bandwidth Saturation Pulse

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Introduction

- Criteria for desirable saturation profile are flat passband and sharp profile with minimum peak $rf(t)$ amplitude. Design parameters for RF pulses include passband δ_1 and stopband δ_2 ripple tolerances and time-bandwidth product tb [1].
- The well-known Shinnar-Le Roux (SLR) RF pulse design technique is a transform that relates magnetization profile to two polynomials A_N and B_N [2-4]. B_N is traditionally obtained by digital filter design techniques.
- A conventional approach (for minimum-peak $rf(t)$) is to design a maximum-phase B_N, factor B_N to obtain its roots, then combinatorially search by root inversion [5] over all possible phase patterns. But this conventional method is limited to $tb\approx 18$ before number of combinations becomes prohibitive.
- Our method uses convex optimization to design globally minimum peak RF saturation pulse. The use of autocorrelation matrix of B_N overcomes a non-convex passband ripple constraint. But the rank of this autocorrelation matrix has to be 1, which again is non-convex. Our solution is to use *convex iteration.*

Methods

- For saturation pulses, the RF pulse $rf(t)$ and B_N are very much alike.
- Our technique: use optimization to generate a minimum peak B_N, then the saturation profile $|M_z|$ and RF pulse $rf(t)$ are found via SLR transform as in the conventional method.

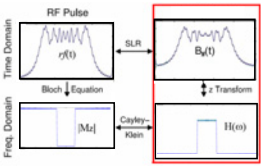

Fig. 1
The four black boxes relate functions involved in conventional RF pulse design. Our technique concentrates mainly within the red box. Figure adapted from Schulte.

- Our problem statement:

$$\begin{array}{ll} \text{minimize} & \|B_N\|_\infty \\ \text{subject to} & 1-\delta_1 \le |H(\omega)| \le 1+\delta_1\,, \ \ \omega\in[0,\omega_p] \\ & |H(\omega)| \le \delta_2\,, \qquad\qquad \omega\in[\omega_s\,,\,\pi] \end{array} \qquad \textbf{Eq. 1}$$

where $H(\omega)$ is the Fourier Transform of B_N. But this problem statement is nonconvex (because of the left hand side of the first constraint).

- So instead, define an autocorrelation matrix of B_N as $G = B_N B_N^{\mathrm{H}}$ where G is positive semidefinite with rank 1.
- Summing along each of $2N-1$ subdiagonals produces entries of the autocorrelation function r of B_N where (Fig. 2)

$$r(i) = \sum_{j=0}^{N-1} B_N(j)B_N(j+i)^* \quad \text{where} \quad r = r_{\mathrm{re}} + i\,r_{\mathrm{im}}$$

Fig. 2
Autocorrelation matrix G is positive semidefinite with rank 1. G is rank 1 because every column is made up of vector B_N shifted by different amount and scaled differently.

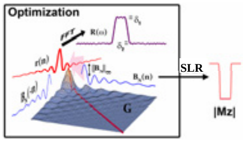

Fig. 3
Relationship of parameters and functions used in optimization. G is composed of products of entries in B_N. The red arrow indicates: summing the main diagonal of G results in $r(0)$. Summing along each subdiagonal of G will give other entires of r. The magnitude spectrum of $r(n)$ will need to meet design specifications. Once an optimal B_N is found, $|M_Z|$ and *rf(t)* can be found using SLR technique.

- To write $r(n)$ in terms of G, define $I_0 = I$, define I_n as a zero matrix having vector **1** along the n^{th} superdiagonal when n is positive or **1** along the n^{th} subdiagonal when n is negative.
- An important property of G: its main diagonal holds squared absolute entries of B_N (red circles in Fig. 2). Minimizing $\|B_N\|_\infty$ is therefore equivalent to minimizing $\|\mathrm{diag}(G)\|_\infty$.
- By spectral factorization [6], a problem equivalent to Eq.1 is

$$\begin{array}{lll} \underset{G,r}{\text{minimize}} & \|\mathrm{diag}(G)\|_\infty & \\ \text{subject to} & R(\omega) = r_{\mathrm{re}}(0) + 2\sum_{n=1}^{N-1}\left(r_{\mathrm{re}}\cos(\omega n) + r_{\mathrm{im}}\sin(\omega n)\right) & \\ & (1-\delta_1)^2 \le R(\omega) \le (1+\delta_1)^2, & \omega\in[-\omega_p\,,\,\omega_p] \\ & R(\omega) \le \delta_2^2\,, & \pm\omega\in[\omega_s\,,\,\pi] \\ & R(\omega)\ge 0\,, & \omega\in[-\pi\,,\,\pi] \\ & r(n) = \mathrm{trace}(I_n^{\mathrm{T}}G)\,, & n=0\ldots N-1 \\ & G\succeq 0 & \\ & \mathrm{rank}(G) = 1 & \end{array} \qquad \textbf{Eq. 2}$$

- Figure 3 shows the relationship of these parameters.
- Excepting the rank constraint, this problem is convex. To overcome the rank constraint limitation, the technique *convex iteration* is used and is described as follows.

Convex Iteration for rank constraint

- The idea of *convex iteration* is to rewrite Eq. 2 as a sequence of convex optimization problems by introducing a *direction vector* W:

While $\mathrm{rank}(G) > 1$
{

$$\begin{array}{lll} \underset{G,\ r}{\text{minimize}} & \langle G\,, W\rangle & \\ \text{subject to} & R(\omega) = r_{\mathrm{re}}(0) + 2\sum_{n=1}^{N-1}\left(r_{\mathrm{re}}(n)\cos(\omega n) + r_{\mathrm{im}}(n)\sin(\omega n)\right) & \\ & (1-\delta_1)^2 \le R(\omega) \le (1+\delta_1)^2 & \omega\in[-\omega_p,\omega_p] \\ & R(\omega) \le \delta_2^2 & \pm\omega\in[\omega_s\,,\pi] \\ & R(\omega)\ge 0 & \omega\in[-\pi,\pi] \\ & r(n) = \mathrm{trace}(I_n^{\mathrm{T}}G) & n=0\ldots N-1 \\ & G\succeq 0 & \\ & \mathrm{diag}(G)\le h_{\max}^2 & \end{array} \qquad \textbf{Eq. 3}$$

Update W

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- where $h_{\max}$ is the desired impulse response peak magnitude. This constraint on $\mathrm{diag}(G)$ replaces the objective from Eq. 2.
- This sequence of convex optimization problems is executed sequentially beginning $W = I$.
- After each iteration, the direction vector W is updated. If $\mathrm{rank}(G) > 1$, then the next iteration will be computed using an updated W. This process is repeated until $\mathrm{rank}(G) = 1$.

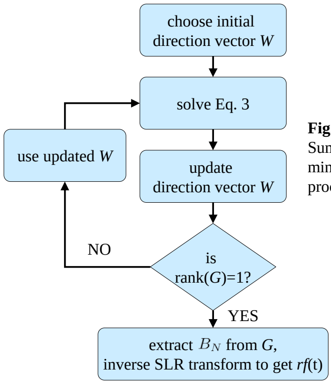

Fig. 4
Summary of proposed minimum peak RF design procedure.

Choosing direction vector W

- Eigenvectors of W are chosen to be identical to optimal G from Eq. 3. Let the eigendecomposition of G be $G = Q\Lambda Q^{\mathrm{T}}$
- The direction vector W is updated by setting its eigenvectors as Q and by setting its eigenvalues, whose locations correspond to the smallest $N-1$ eigenvalues of G, to be 1; and 0 elsewhere. The idea is, on the next iteration, the smallest eigenvalues of G get minimized. We choose

$$W = Q\Phi Q^{\mathrm{T}}\,, \qquad \Phi = \begin{bmatrix} 0 & & & & \mathbf{0} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ \mathbf{0} & & & & 1 \end{bmatrix}$$

- This way, $\langle G\,, W\rangle = \mathrm{tr}(GW) = \mathrm{tr}(Q\Lambda\Phi Q^T) = \sum_{i=2}^{N}\Lambda_{ii}$

 The vector inner product is a sum of the $N-1$ smallest eigenvalues of G.
- Rank of G must equal 1 when those $N-1$ eigenvalues vanish because $\mathrm{rank}\,G = \mathrm{rank}\,\Lambda$
- Once a rank-one matrix G is found, B_N can be obtained by normalizing the first column of G (Fig. 2). Using B_N, RF waveform $rf(t)$ can be computed using inverse SLR transform.
- The entire design procedure is summarized in Fig. 4.

Results

- Figure 5 shows $rf(t)$ using the proposed optimization, zero-flipping root-inversion search, and minimum phase design techniques. Here, time-bandwidth product ($tb = 14$) is small enough to compare our result with exhaustive root-inversion search.

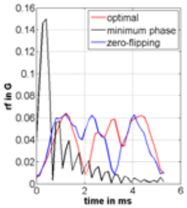

Fig. 5
RF pulses for *tb*=14, $\delta_1 = \delta_2 = 0.13\%$.

- Figure 6 shows $rf(t)$ with larger $tb = 32$. The slice profile (Fig. 6a) of our optimal design is very close to the minimum phase design, while the peak RF (Fig. 6b) power using our technique is much lower. Bloch simulation (Fig. 6c) of the optimal least peak RF is shown for verification.

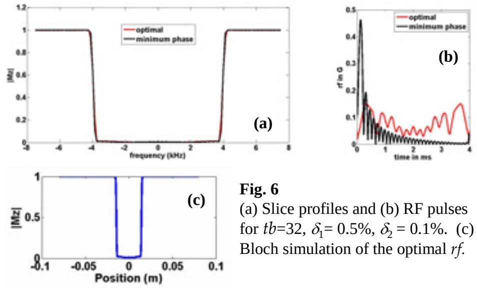

Fig. 6
(a) Slice profiles and (b) RF pulses for *tb*=32, $\delta_1 = 0.5\%$, $\delta_2 = 0.1\%$. (c) Bloch simulation of the optimal *rf*.

Discussion

We presented a method to design an RF saturation pulse having globally minimum peak RF power without the need for exhaustive root-inversion search. To formulate the problem as convex, we employ the tricks of autocorrelation matrix and *convex iteration*.

1.Tuan-Khanh *et.al.* MRM,43(1),2000. 2.Shinnar *et.al.* MRM,12(1),1989. 3.LeRoux *et.al.* JMR,8,1988. 4.Pauly *et.al.* IEEE Trans Med Imaging,10,1991. 5.Schulte *et.al.* JMR,166(1),2004. 6.Wu *et.al.* Applied and Computational Control,vol.1.Birkhuser Boston,1999.

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