# Compressed sensing with contiguous Fourier measurements

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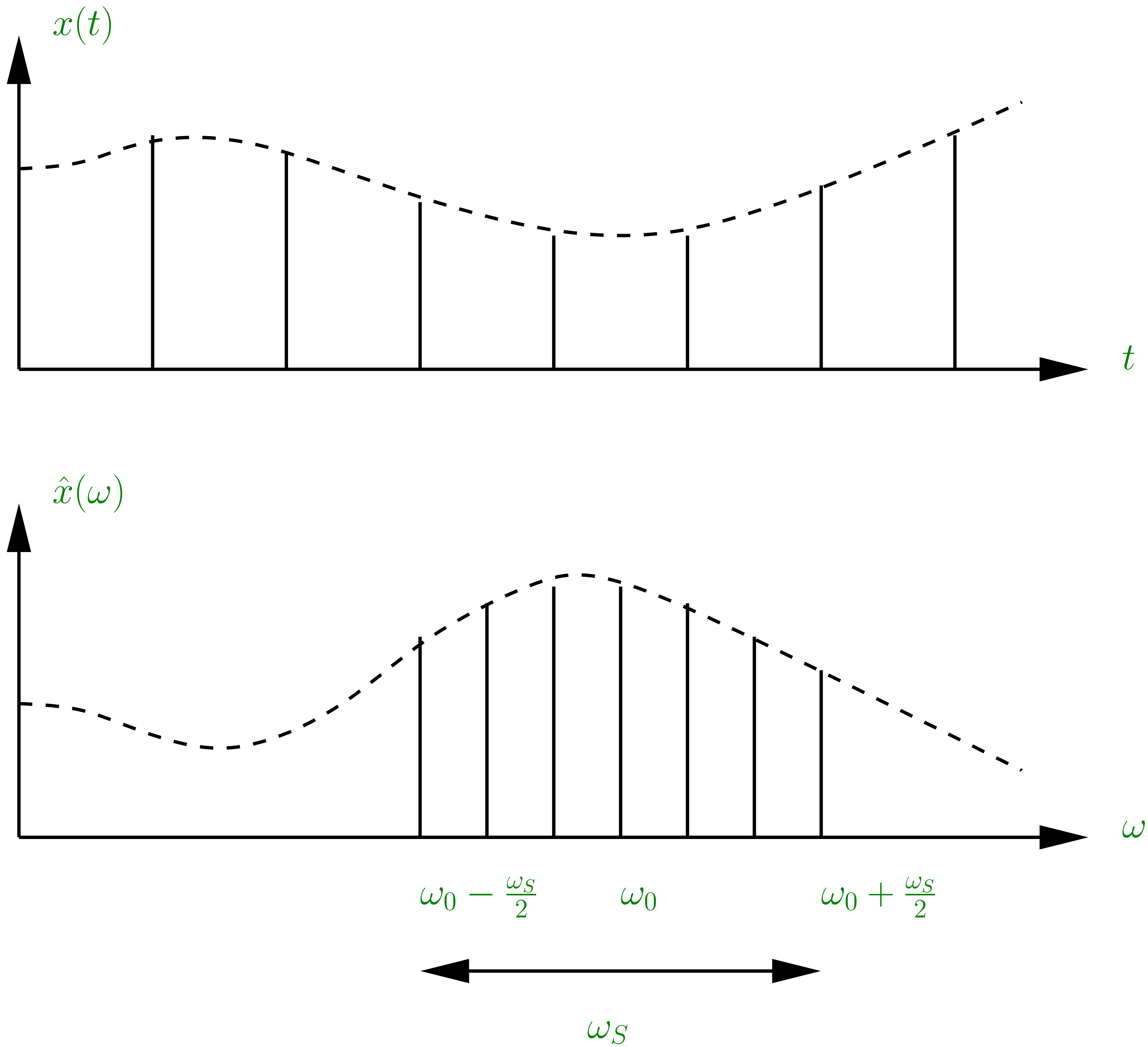
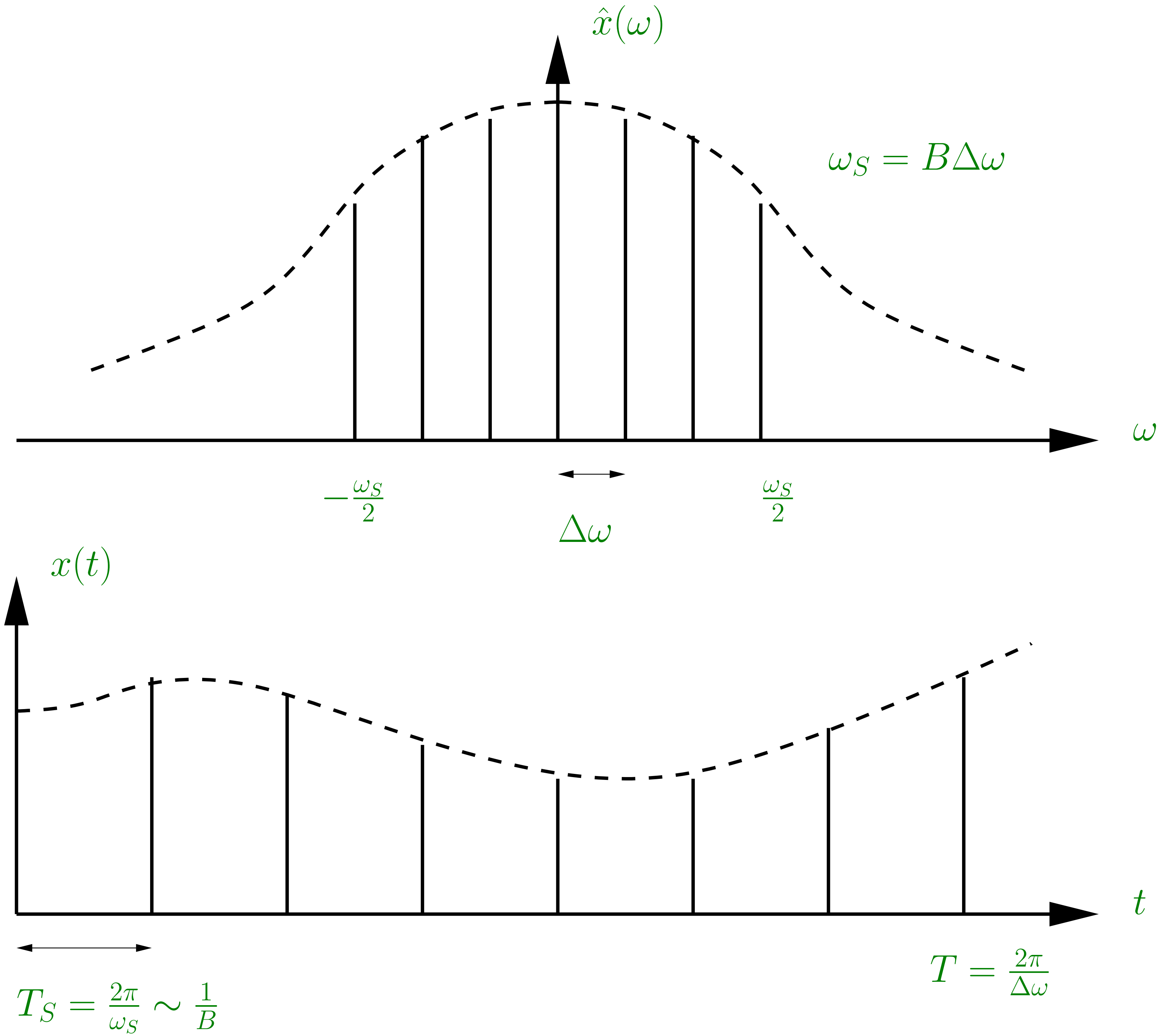
# Outline

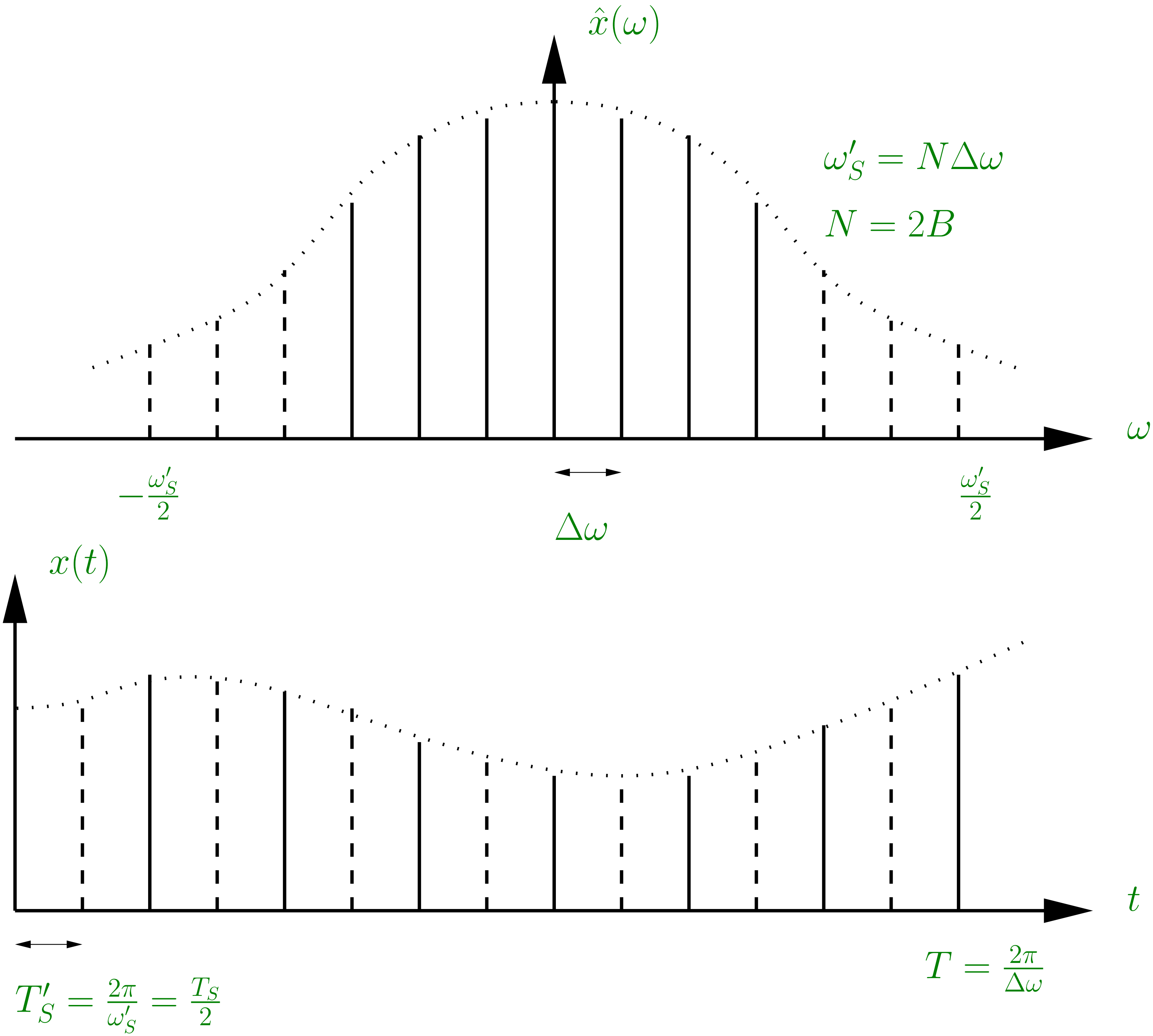
1. studied problem
2. link with Compressed Sensing
3. numerical illustration
4. case of low frequency signals

# Problem studied

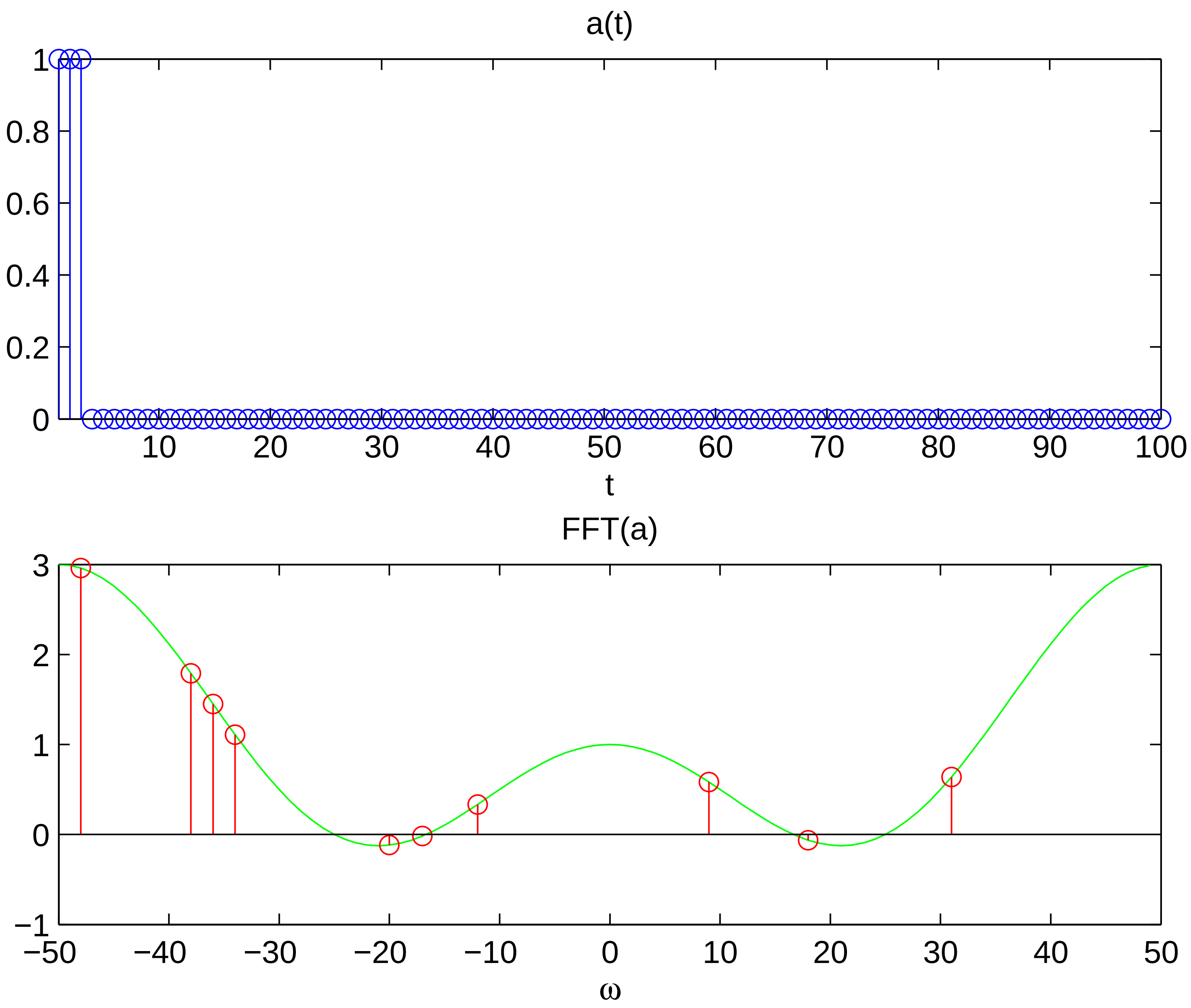




$\hat{x}(\omega)$
$\omega_S = B\Delta\omega$
$\omega$
$-\frac{\omega_S}{2}$
$\Delta\omega$
$\frac{\omega_S}{2}$
$x(t)$
$t$
$T = \frac{2\pi}{\Delta\omega}$
$T_S = \frac{2\pi}{\omega_S} \sim \frac{1}{B}$


$\hat{x}(\omega)$
$\omega'_S = N\Delta\omega$
$N = 2B$
$\omega$
$-\frac{\omega'_S}{2}$
$\frac{\omega'_S}{2}$
$\Delta\omega$
$x(t)$
$t$
$T = \frac{2\pi}{\Delta\omega}$
$T'_S = \frac{2\pi}{\omega'_S} = \frac{T_S}{2}$

# Link with Compressed Sensing (Donoho, Candès)



## $l^1$ minimization problem

$x \approx a$ solution of:

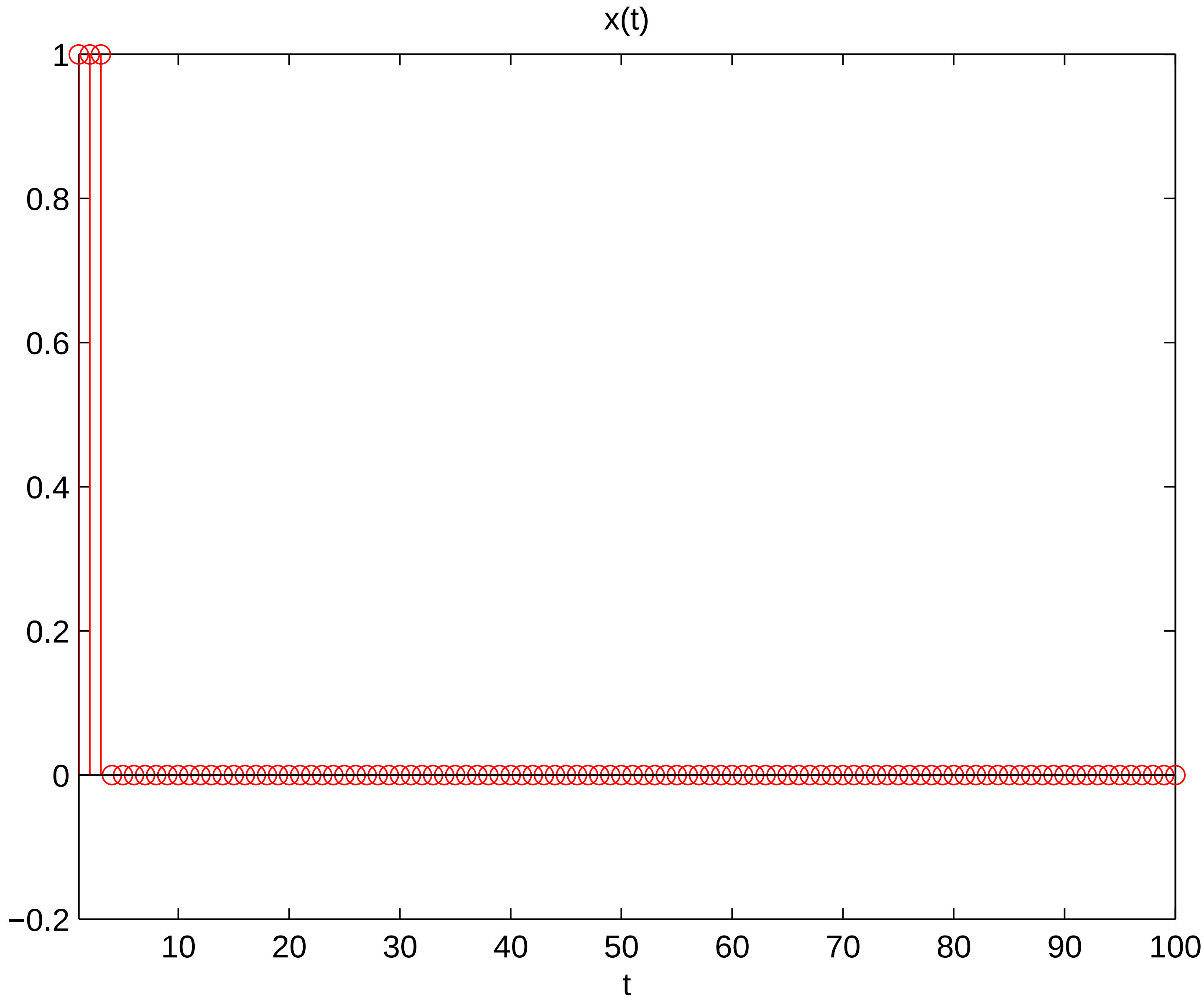
$$\min ||x||_{l^1} \text{ such that } \hat{x}(\omega) = \hat{a}(\omega),$$

$$\omega \in \Omega_B = \{\omega_1, \omega_2, \cdots, \omega_B\}$$

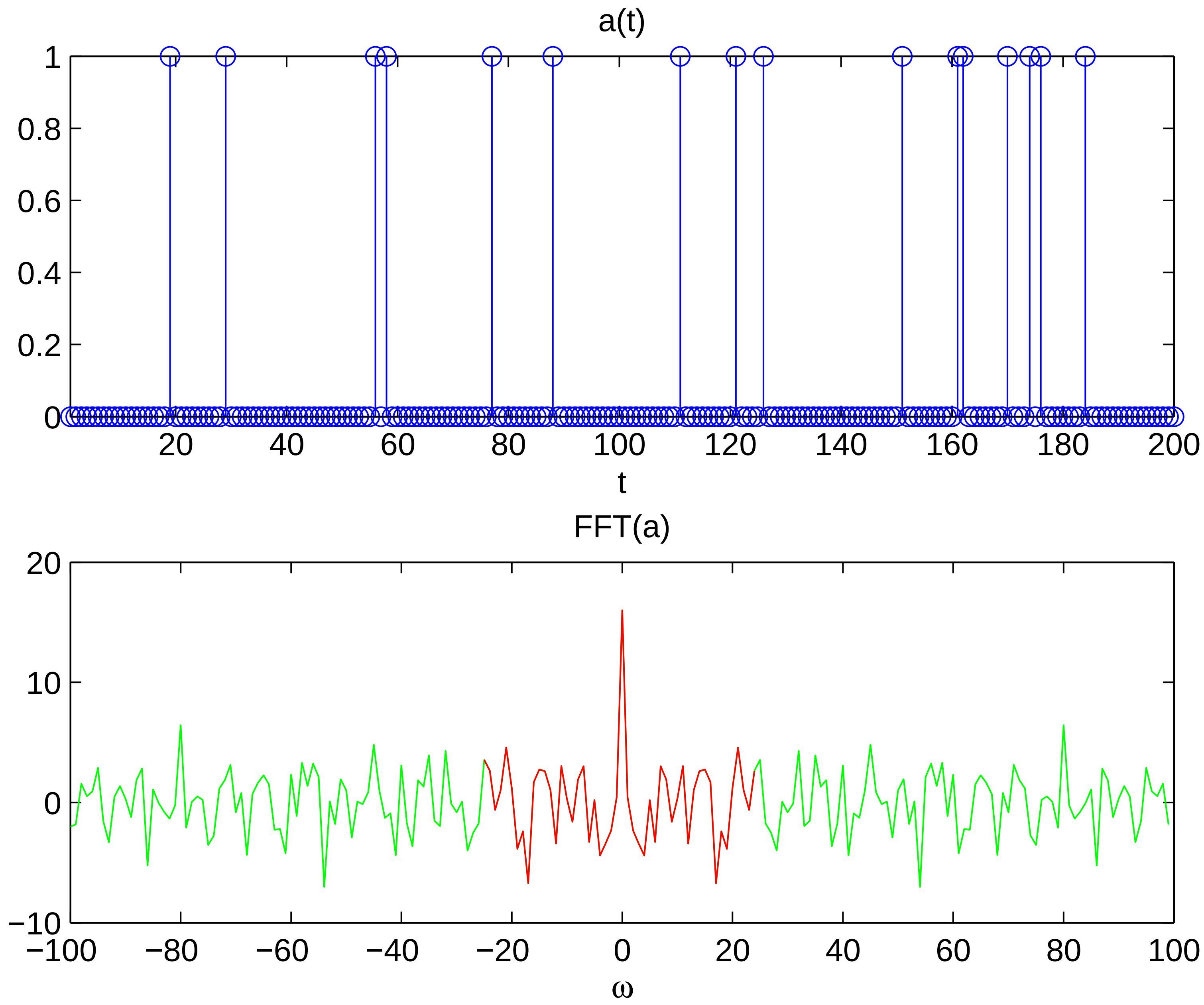
Works if

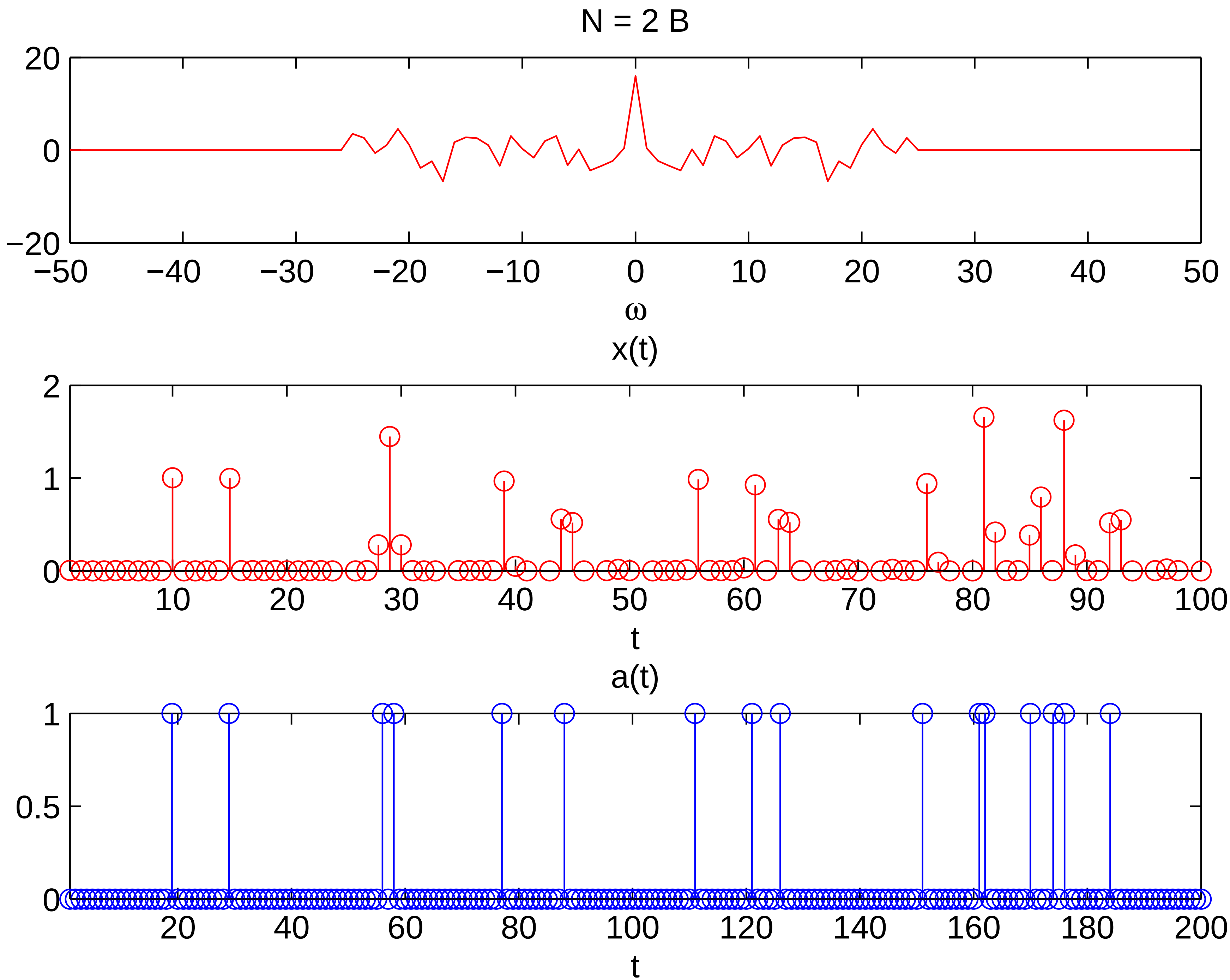
$$B > CS \log N$$

- $B$ number of measurements,
- $S$ sparsity=number of spikes,
- $N$ length of the signal

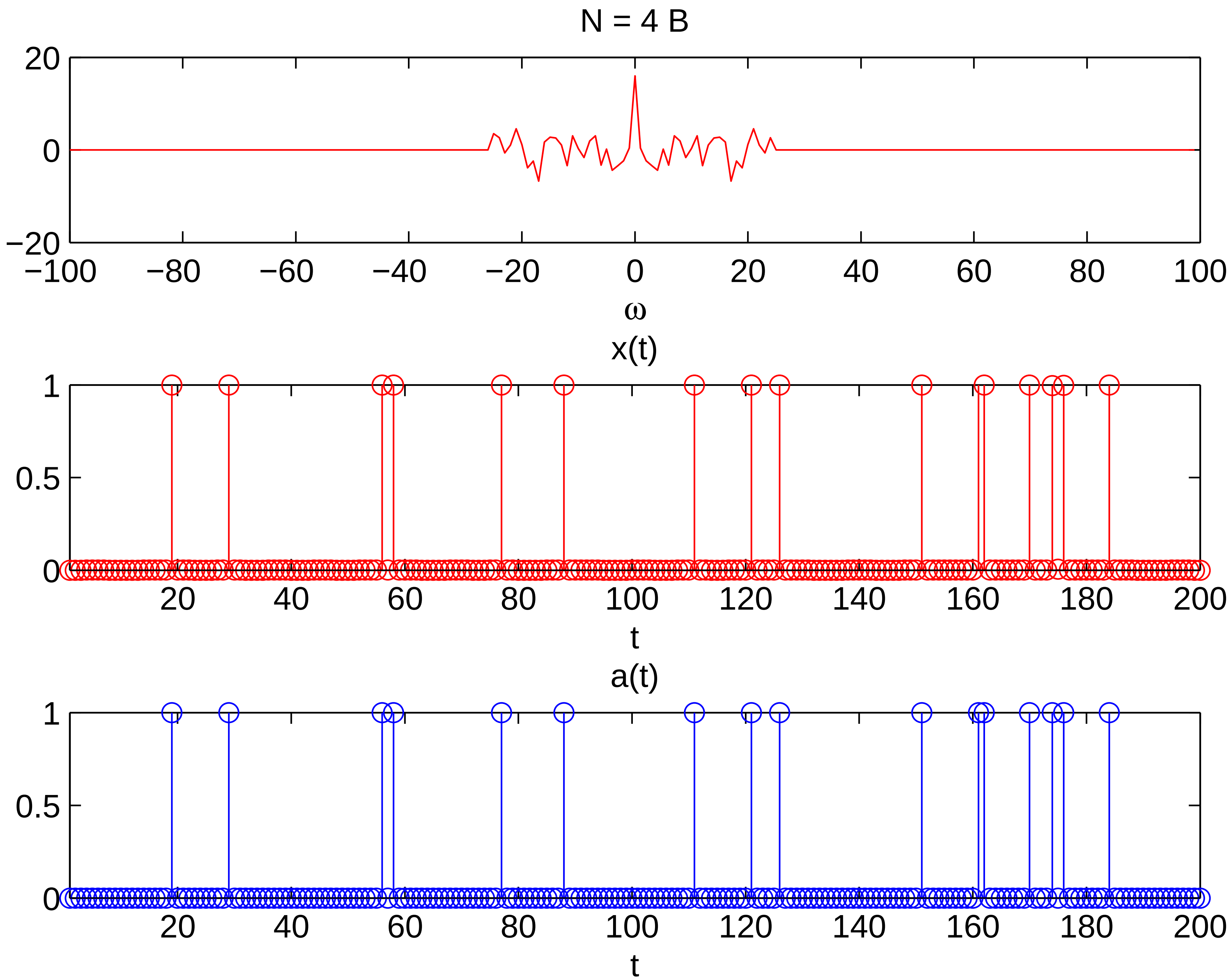

x(t)
1
0.8
0.6
0.4
0.2
0
−0.2
10
20
30
40
50
60
70
80
90
100
t

# Application to contiguous measurements

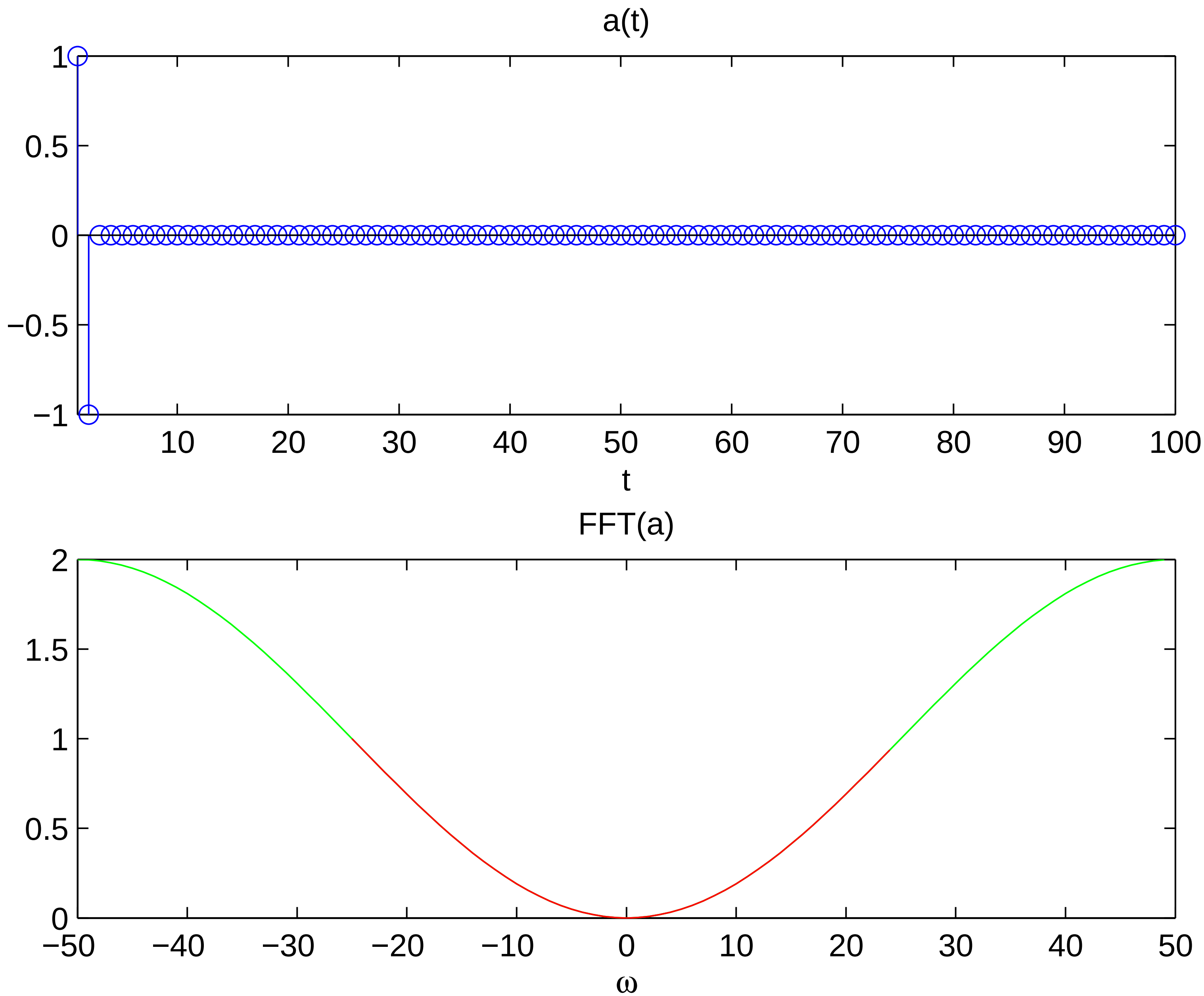


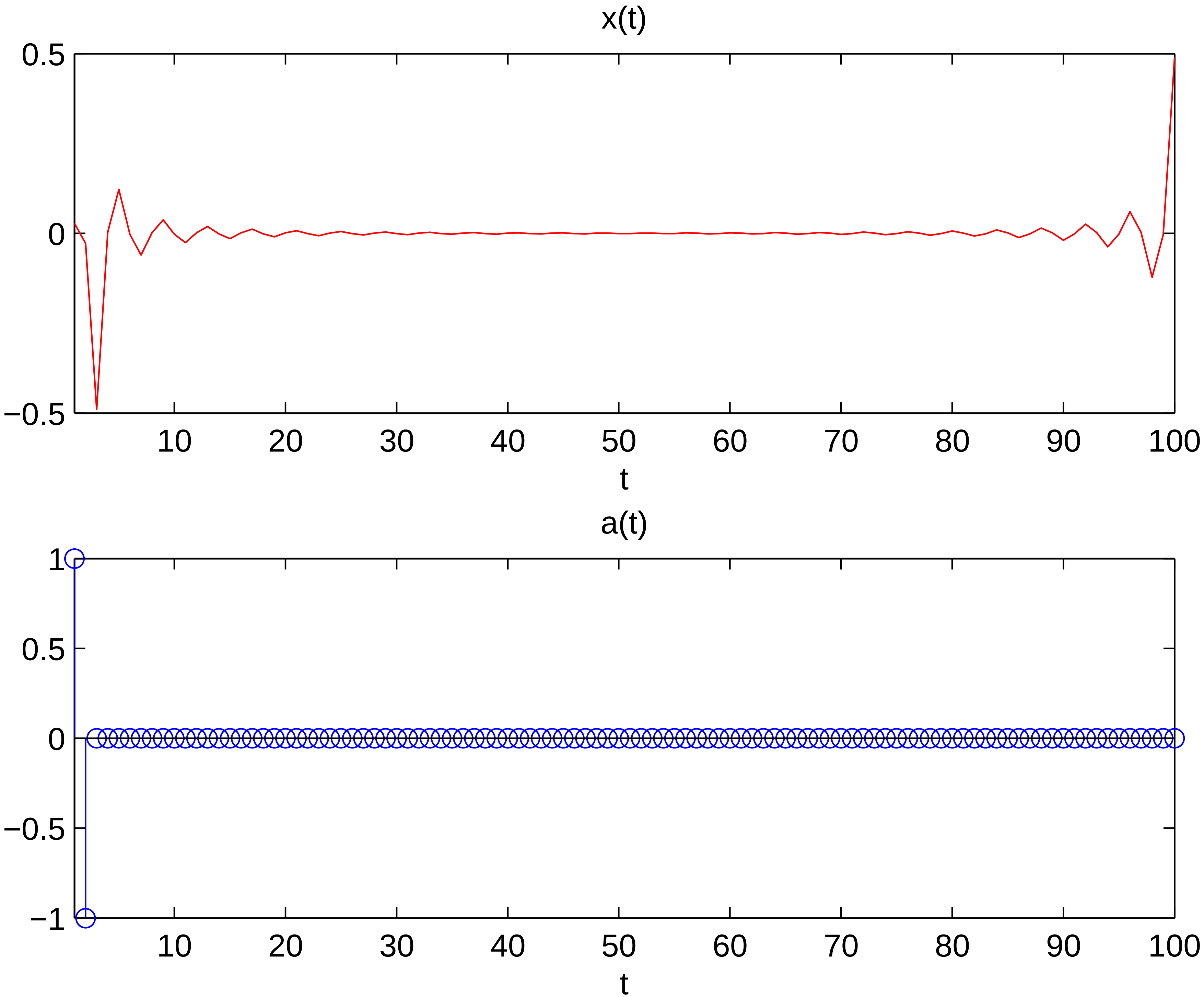


N = 2 B
20
0
−20
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
x(t)
2
1
0
10
20
30
40
50
60
70
80
90
100
t
a(t)
1
0.5
0
20
40
60
80
100
120
140
160
180
200
t

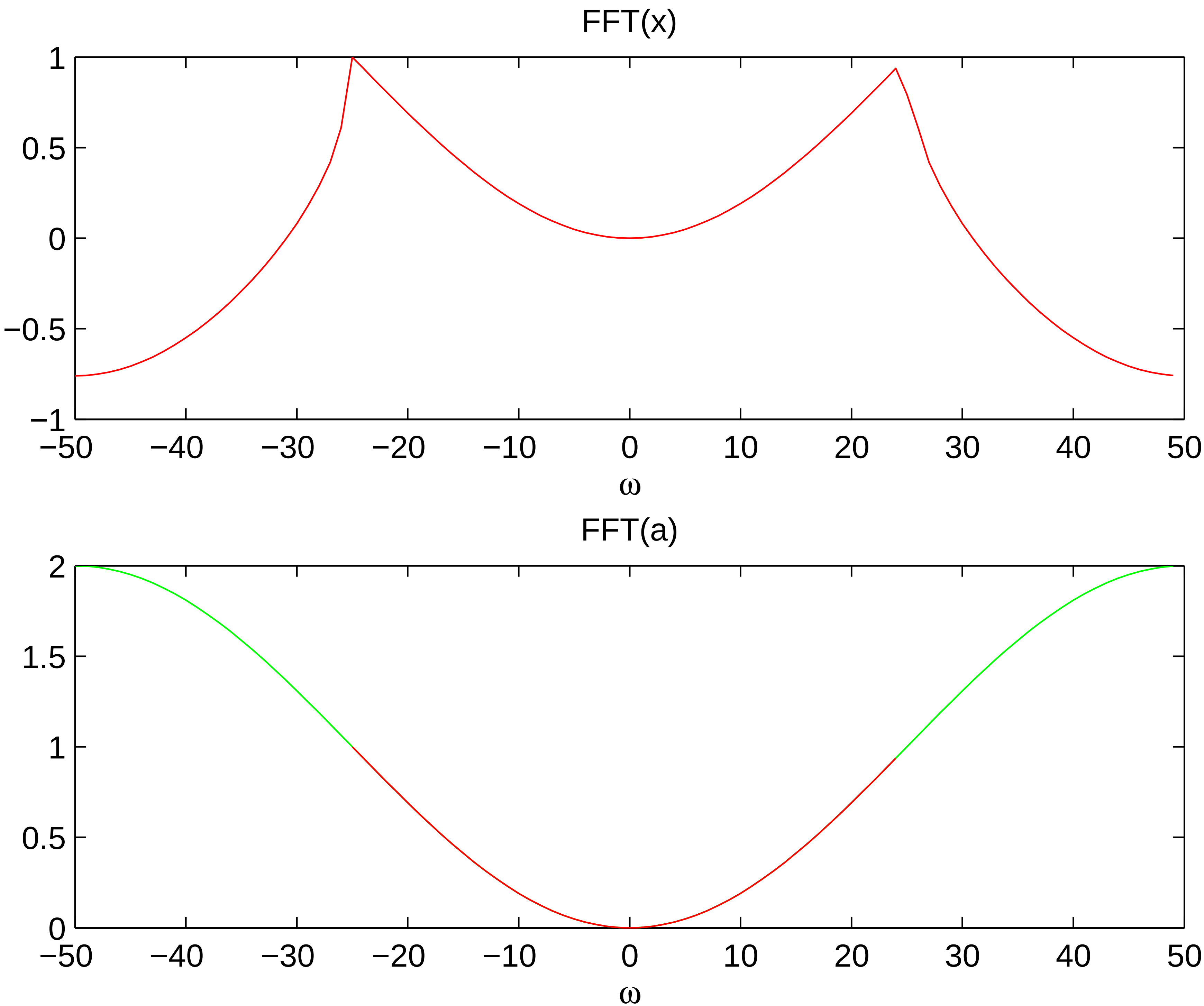

N = 4 B
20
0
−20
−100
−80
−60
−40
−20
0
20
40
60
80
100
ω
x(t)
1
0.5
0
20
40
60
80
100
120
140
160
180
200
t
a(t)
1
0.5
0
20
40
60
80
100
120
140
160
180
200
t

# Case of high frequency signals





x(t)
0.5
0
−0.5
10
20
30
40
50
60
70
80
90
100
t
a(t)
1
0.5
0
−0.5
−1
10
20
30
40
50
60
70
80
90
100
t


FFT(x)
1
0.5
0
−0.5
−1
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
FFT(a)
2
1.5
1
0.5
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω

# Filter

We want $x(t) \approx a(t)$ when $a$ high frequency

Idea : two steps

1. find intermediate $y$ not high frequency,
2. deduce $x$.

# Filter

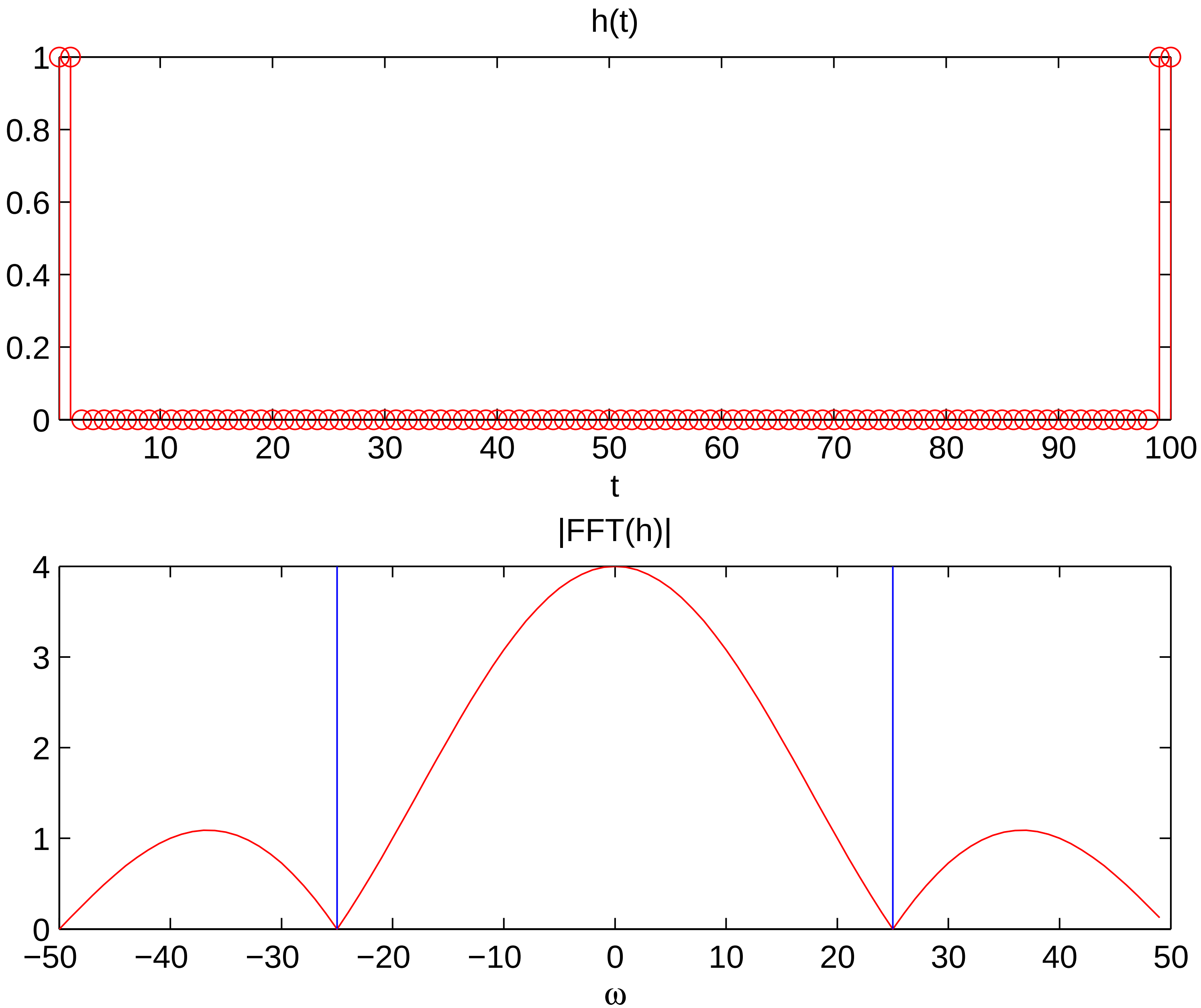
Two steps:

1. find $y = x * h$, $h$ well chosen,
2. deduce $x$.

$h$?

$$y = x * h \Longleftrightarrow \hat{y} = \hat{x}\hat{h}$$

# Sinc in the case $N = 2B$



# Two steps :

Data= $\hat{a}|_{\Omega_B}$

Usual CS:

$$\min ||x||_{l^1} \text{ such that } \hat{x}|_{\Omega_B} = \hat{a}|_{\Omega_B}$$

For high frequency signals:

1.

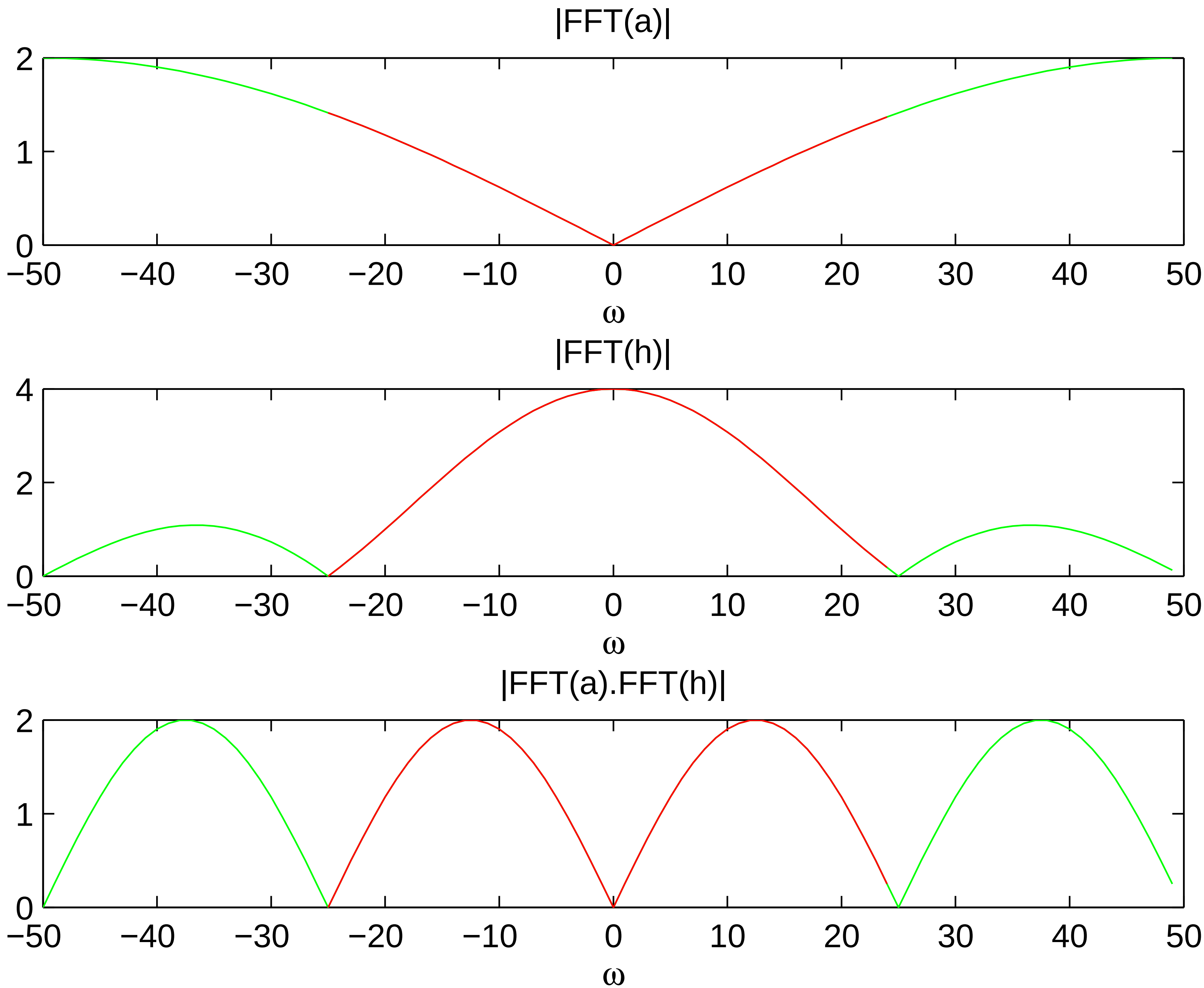
$$\min ||y||_{l^1} \text{ such that } \hat{y}|_{\Omega_B} = (\hat{a}\hat{h})|_{\Omega_B}$$

$$\hookrightarrow y \approx a * h \text{ and } \hat{y} \approx \hat{a}\hat{h}$$

2.

$$\min ||z||_{l^1} \text{ such that } ||\hat{h}\hat{z} - \hat{y}||_{l^2} \leq \epsilon ||\hat{y}||_{l^2}$$

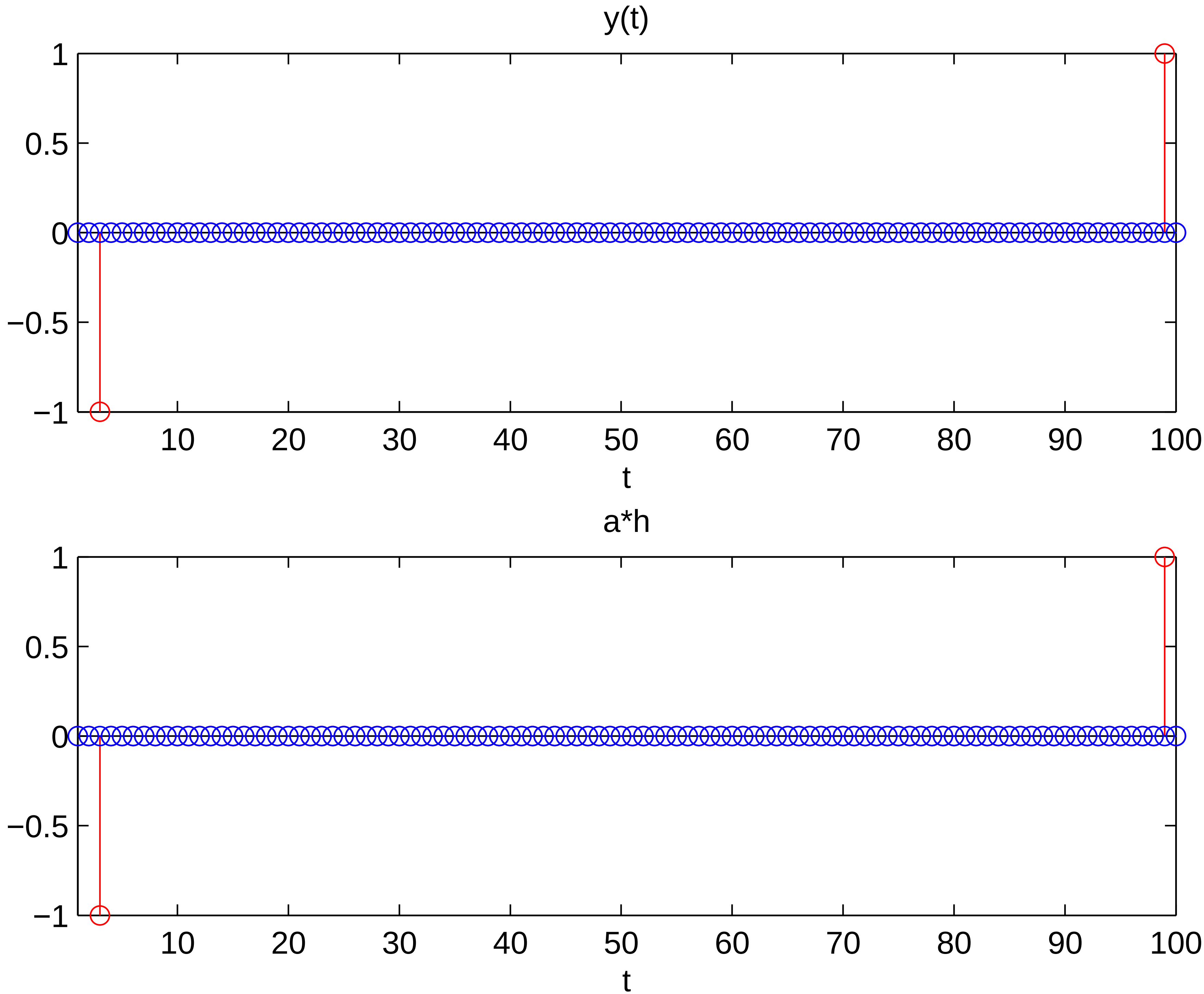
$$\hookrightarrow z \approx a$$


|FFT(a)|
2
1
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
|FFT(h)|
4
2
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
|FFT(a).FFT(h)|
2
1
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω

# First step

$$\min ||y||_{l^1} \text{ such that } \hat{y}|_{\Omega_B} = (\hat{a}\hat{h})|_{\Omega_B}$$

$$\hookrightarrow y \approx a * h \text{ and } \hat{y} \approx \hat{a}\hat{h}$$



y(t)
1
0.5
0
−0.5
−1
10
20
30
40
50
60
70
80
90
100
t
a*h
1
0.5
0
−0.5
−1
10
20
30
40
50
60
70
80
90
100
t

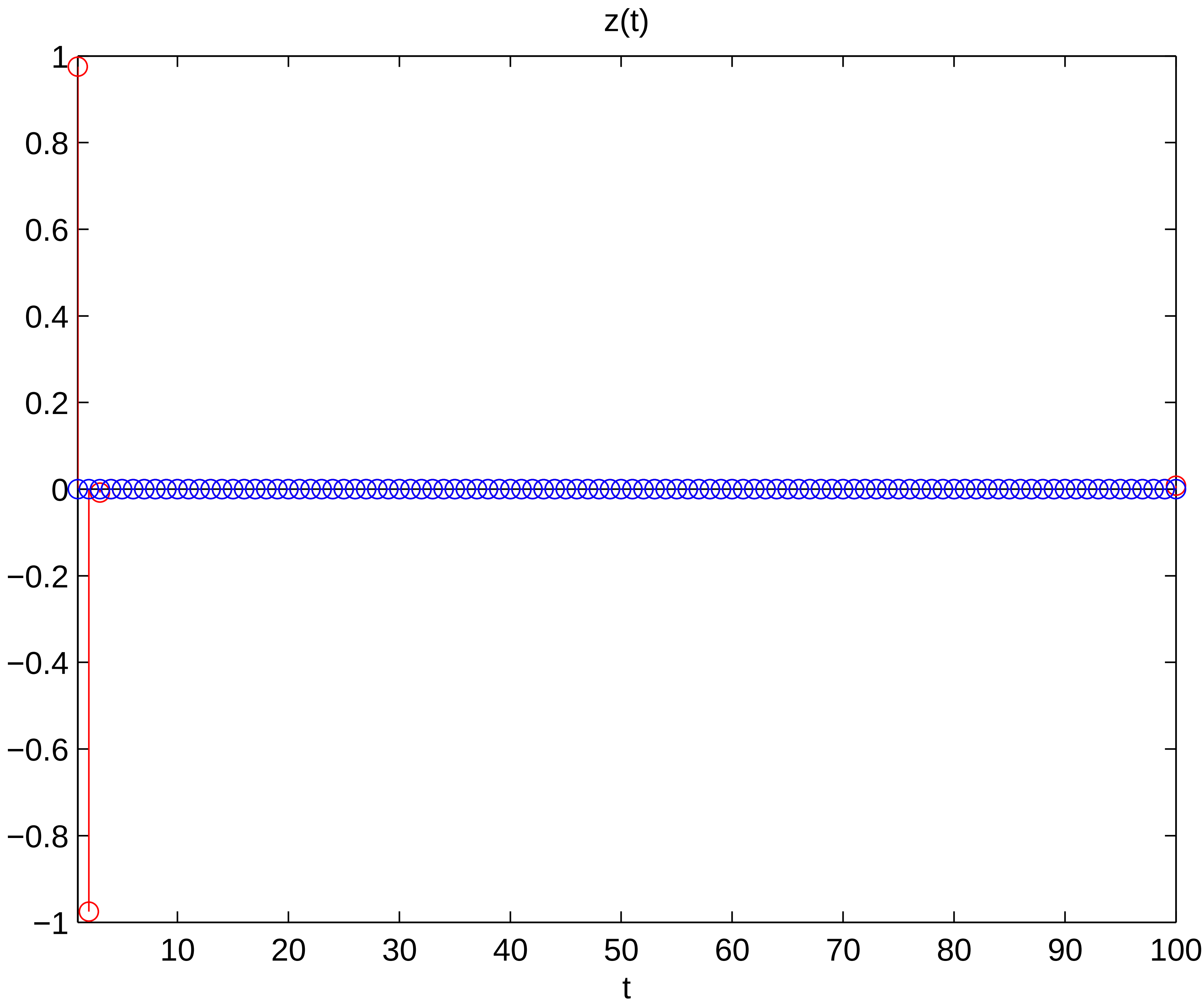
# Second step

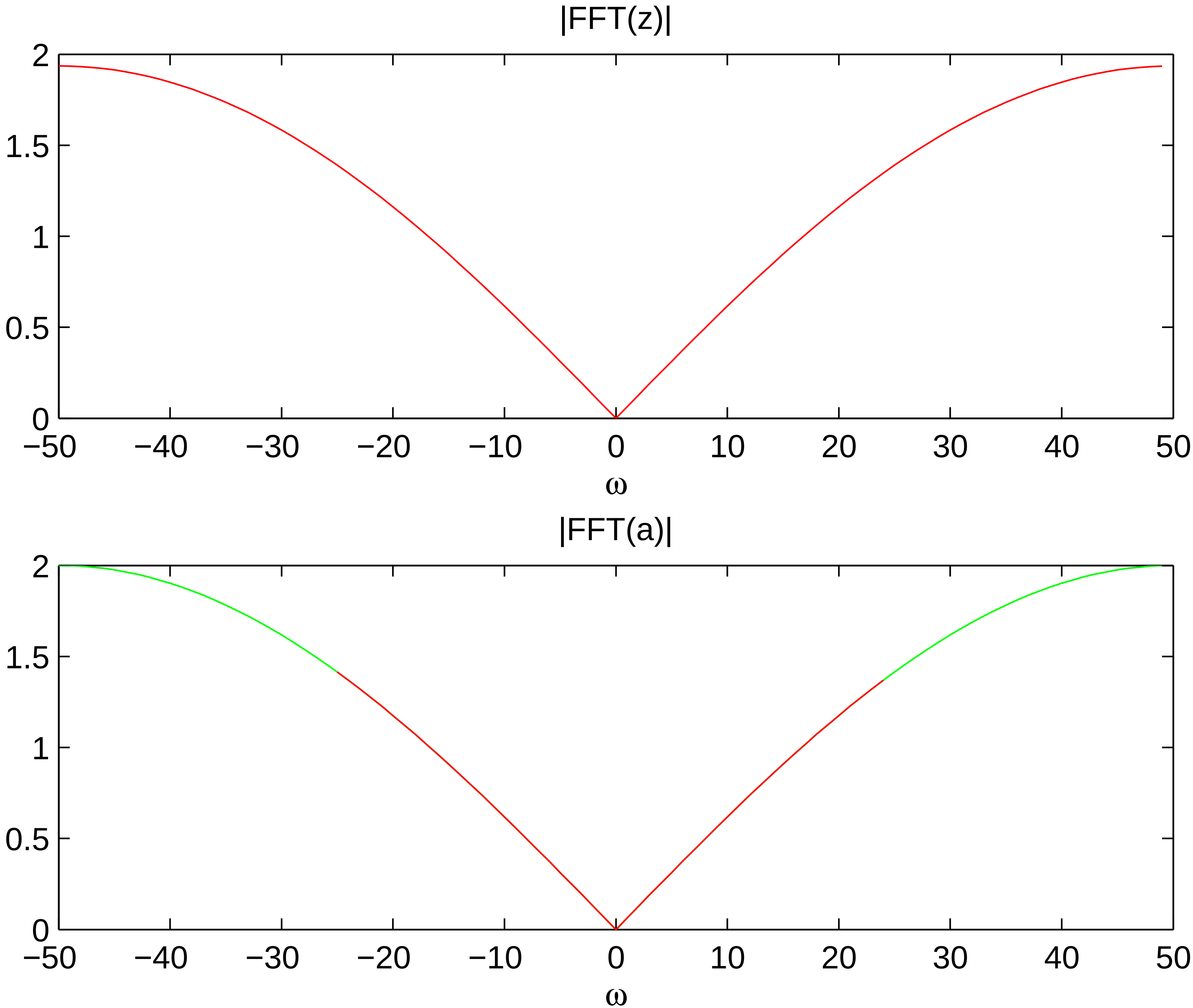
We have

$$\hat{y} \approx \hat{a}\hat{h}$$

$$\min ||z||_{l^1} \text{ such that } ||\hat{h}\hat{z} - \hat{y}||_{l^2} \leq \epsilon ||\hat{y}||_{l^2}$$

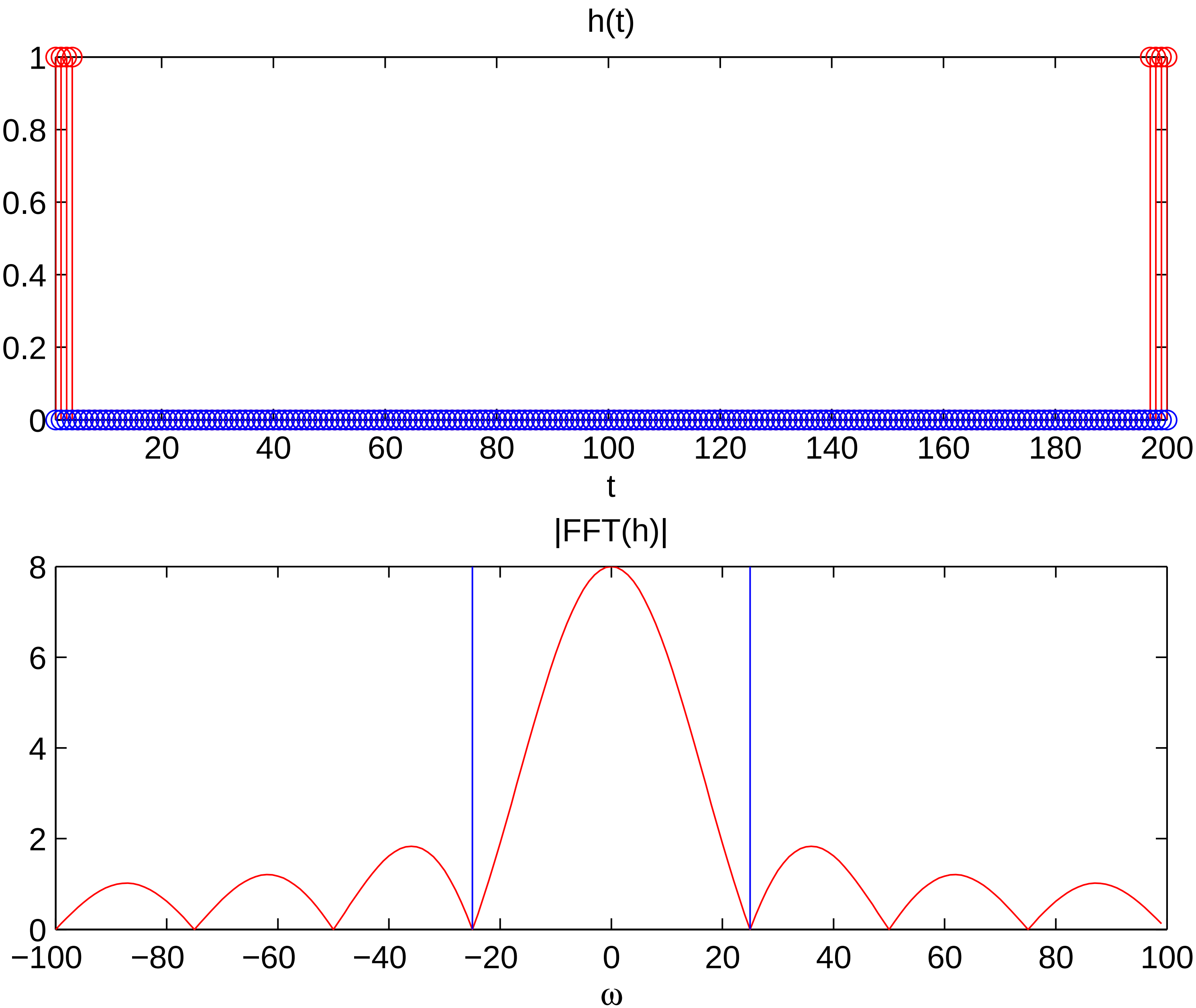
$$\hookrightarrow z \approx a$$


z(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
10
20
30
40
50
60
70
80
90
100
t

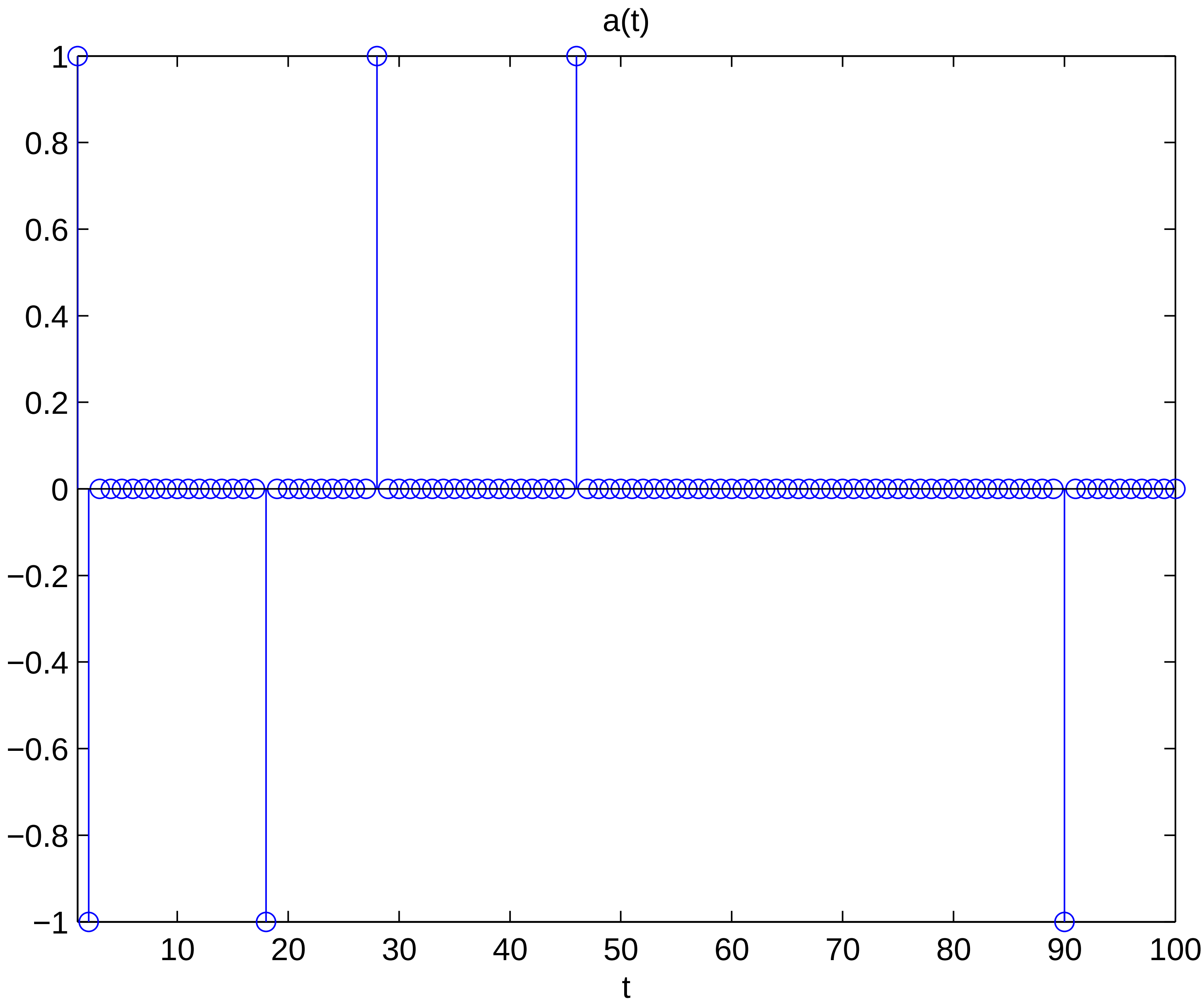


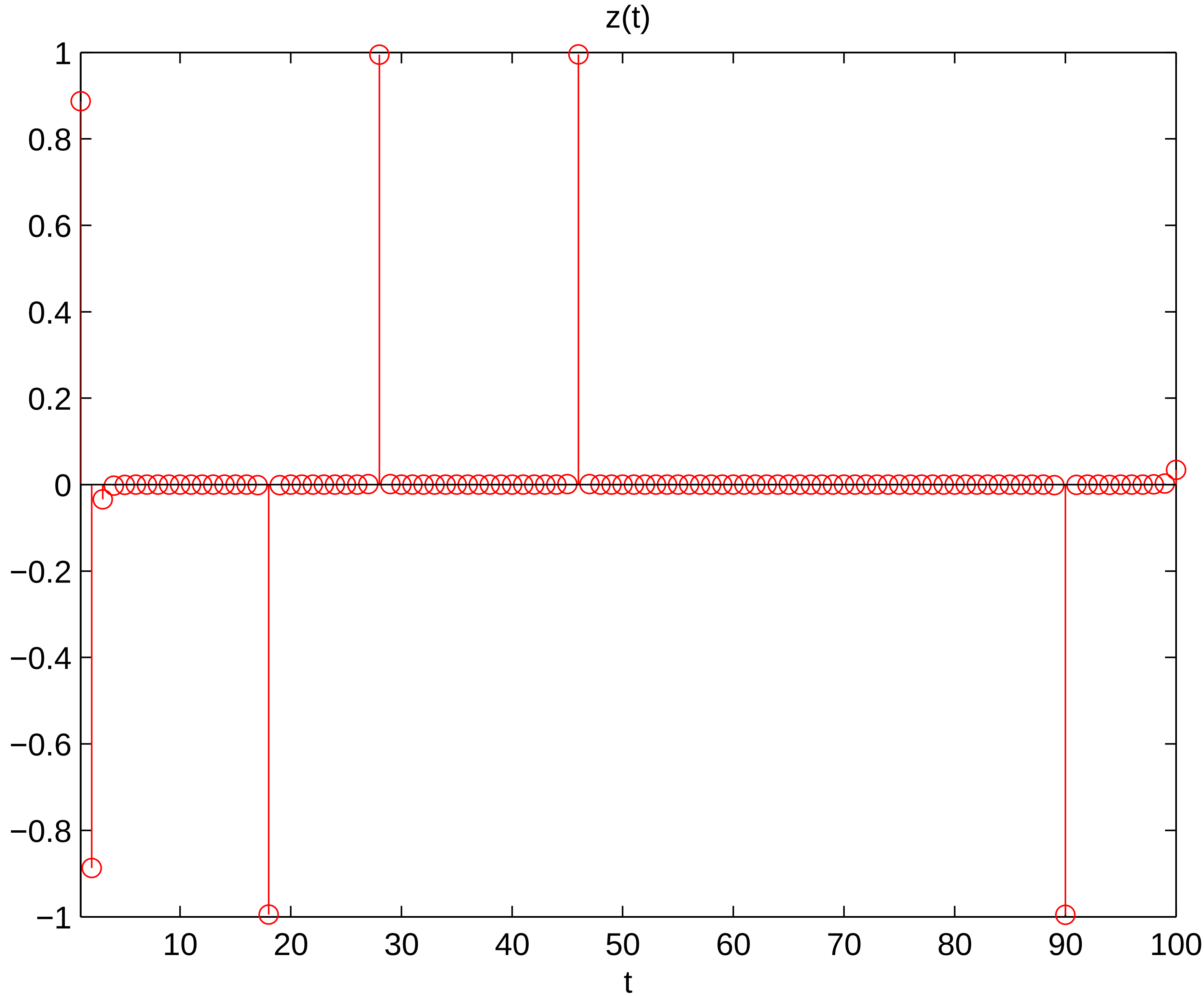
|FFT(z)|
2
1.5
1
0.5
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
|FFT(a)|
2
1.5
1
0.5
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω

# Filter in the case $N = 4B$



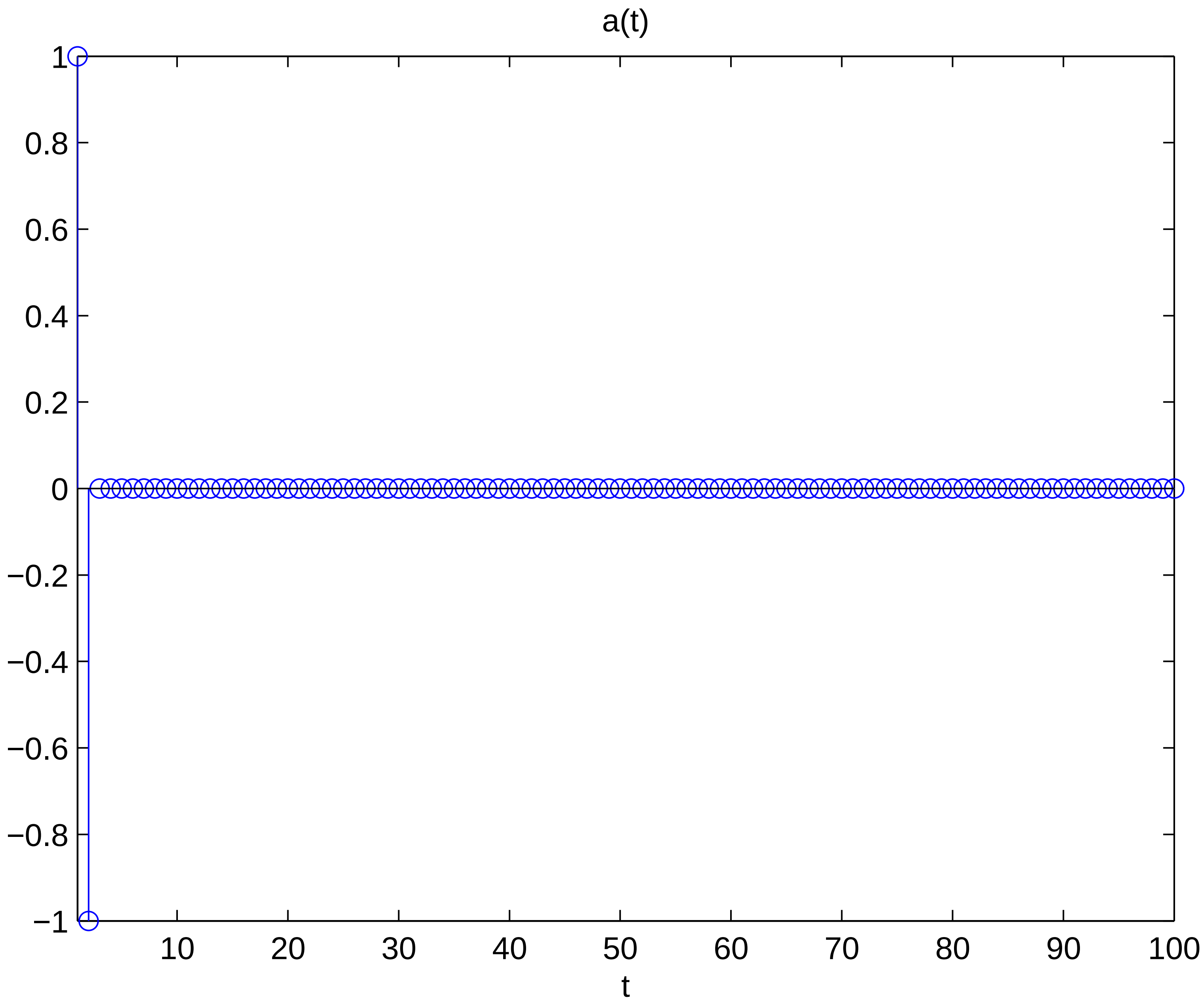
# Perspective : robustness of the filter?

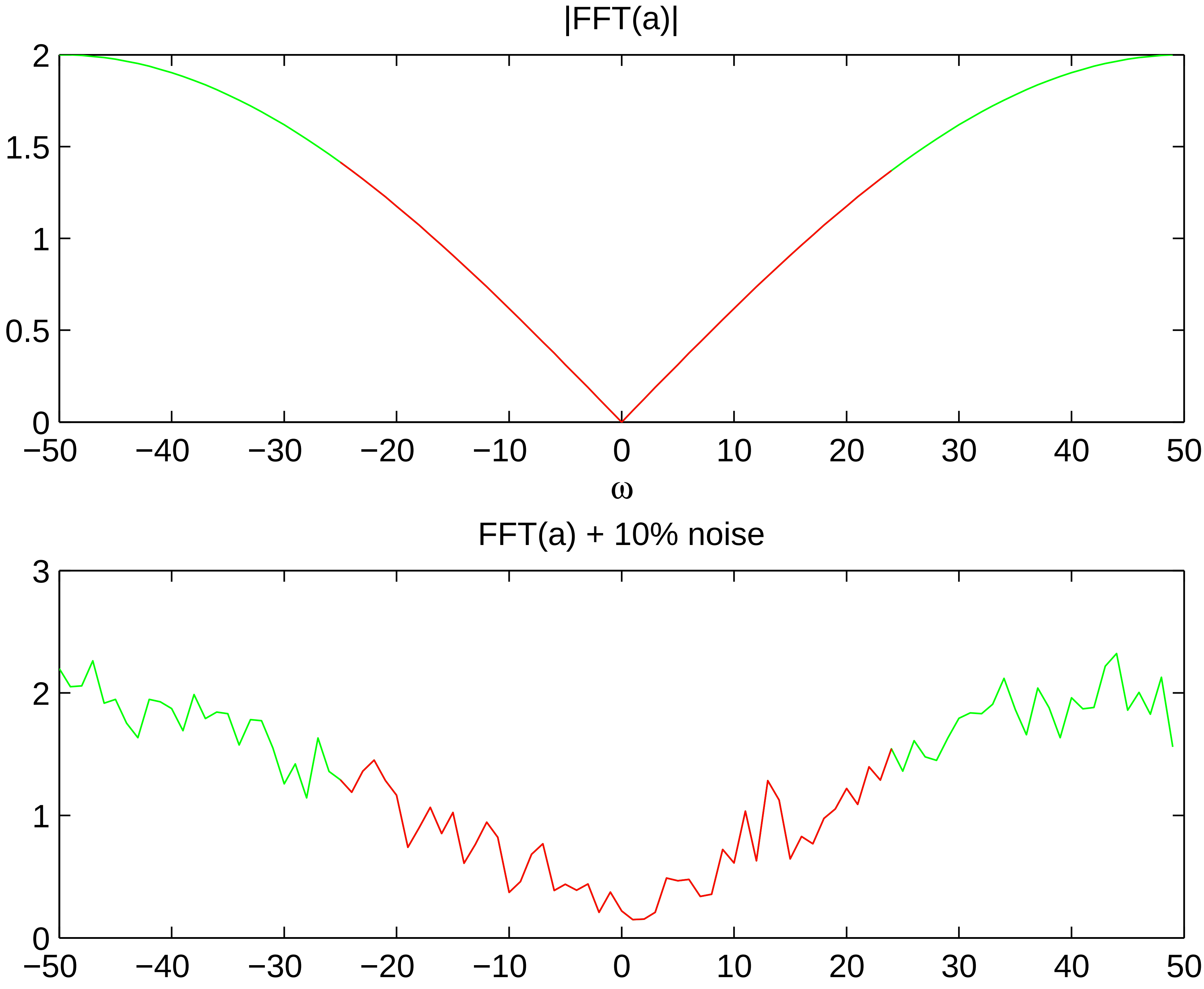




z(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
10
20
30
40
50
60
70
80
90
100
t

# Influence of noise

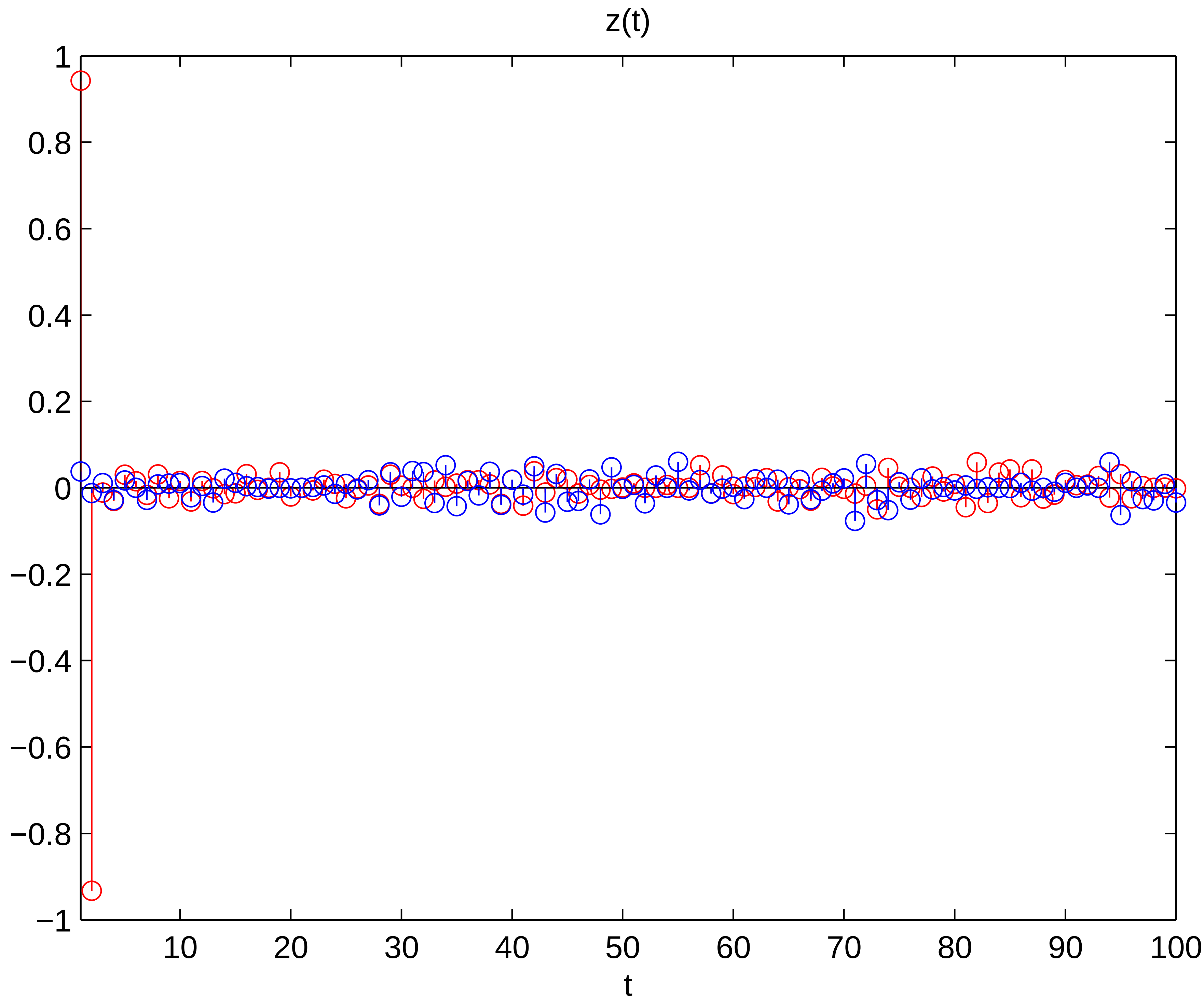




|FFT(a)|
2
1.5
1
0.5
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω
FFT(a) + 10% noise
3
2
1
0
−50
−40
−30
−20
−10
0
10
20
30
40
50
ω

$$\hat{a} \longrightarrow \hat{a} + \sigma \cdot \mathrm{randn}$$

$$\sigma = \frac{\max |\hat{a}|}{10}$$


z(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
10
20
30
40
50
60
70
80
90
100
t

# General signal