Explicit Sensor Network Localization using Semidefinite Programming and Facial Reduction

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Outline

1. Preliminaries
   - SNL ↔ GR ↔ EDM ↔ SDP
   - Facial Structure of Cones

2. Clique/Facial Reduction (Exploit degeneracy)
   - Basic Single Clique Reduction
   - Two Clique Reduction and EDM DELAYED Completion
   - Completing SNL; DELAYED use of Anchor Locations

   - Clique Unions and Node Absorptions
   - Results (low CPU time; high accuracy)

4. Noisy Data
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4. Noisy Data
Sensor Network Localization, SNL, Problem

SNL - a Fundamental Problem of Distance Geometry;
easy to describe - dates back to Grasssmann 1886

- $n$ ad hoc wireless sensors (nodes) to locate in $\mathbb{R}^r$,  
  ($r$ is embedding dimension; sensors $p_i \in \mathbb{R}^r$, $i \in V := 1, \ldots, n$)

- $m$ of the sensors are anchors, $p_i, i = n - m + 1, \ldots, n$  
  (positions known, using e.g. GPS)

- pairwise distances $D_{ij} = \|p_i - p_j\|^2$, $ij \in E$, are known within radio range $R > 0$

\[
P = \begin{bmatrix}
  p_1^T \\
  \vdots \\
  p_n^T
\end{bmatrix} = \begin{bmatrix}
  X \\
  A
\end{bmatrix} \in \mathbb{R}^{n \times r}
\]
Applications


Untethered micro sensors will go anywhere and measure anything - traffic flow, water level, number of people walking by, temperature. This is developing into something like a nervous system for the earth, a skin for the earth. The world will evolve this way.

Tracking Humans/Animals/Equipment/Weather (smart dust)

- geographic routing; data aggregation; topological control; soil humidity; earthquakes and volcanos; weather and ocean currents.
- military; tracking of goods; vehicle positions; surveillance; random deployment in inaccessible terrains.
Conferences/Journals/Research Groups/Books/Theses/Codes

Citations at end, page 53

- Conference, MELT 2008
- International Journal of Sensor Networks
- Research groups include: CENS at UCLA, Berkeley WEBS,
- recent related theses and books include: [10, 16, 8, 7, 11, 12, 6, 14, 17]
- recent algorithms specific for SNL: [1, 2, 3, 4, 5, 9, 15, 18, 13]
Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i, j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- Realization of $\mathcal{G}$ in $\mathbb{R}^r$: a mapping of node $v_i \rightarrow p_i \in \mathbb{R}^r$ with squared distances given by $\omega$.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance)} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors $p_i, p_j$; anchors correspond to a clique.
Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
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$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors $p_i, p_j$; anchors correspond to a clique.
Sensor Localization Problem/Partial EDM

Sensors □ and Anchors □

Preliminaries
Clique/Facial Reduction (Exploit degeneracy)
Algorithm: Facial Reduct. via Subsp. Inters./DELAYED Compl.
Noisy Data
Summary

SNL <-> GR <-> EDM <-> SDP
Facial Structure of Cones
Connections to Semidefinite Programming (SDP)

\( S^n_+ \), Cone of (symmetric) SDP matrices in \( S^n \); \( x^T Ax \geq 0 \)

inner product \( \langle A, B \rangle = \text{trace } AB \)
Löwner (psd) partial order \( A \succeq B, A \succ B \)

\[ D = \mathcal{K}(B) \in \mathcal{E}^n, \quad B = \mathcal{K}^\dagger(D) \in S^n \cap S_C \text{ (centered } Be = 0) \]

\[ P^T = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}; \quad B := PP^T \in S^n_+; \]
rank \( B = r; \quad D \in \mathcal{E}^n \) be corresponding EDM .

(to \( D \in \mathcal{E}^n \)) \[ D = \left( \|p_i - p_j\|_2^2 \right)_{i,j=1}^n \]
\[ = \left( p_i^T p_i + p_j^T p_j - 2p_i^T p_j \right)_{i,j=1}^n \]
\[ = \text{diag}(B) e^T + e \text{diag}(B)^T - 2B \]
\[ =: \mathcal{D}_e(B) - 2B \]
\[ =: \mathcal{K}(B) \quad \text{(from } B \in S^n_+). \]
Connections to Semidefinite Programming (SDP)

\(S^n_+, \text{Cone of (symmetric) SDP matrices in } S^n; x^T Ax \geq 0\)

inner product \(\langle A, B \rangle = \text{trace } AB\)
Löwner (psd) partial order \(A \succeq B, A \succ B\)

\(D = K(B) \in \mathcal{E}^n, B = K^\dagger(D) \in S^n \cap S_C \text{ (centered } Be = 0)\)

\(P^T = [p_1 \quad p_2 \quad \ldots \quad p_n] \in \mathcal{M}^{r \times n}; B := PP^T \in S^n_+;\)
rank \(B = r; D \in \mathcal{E}^n \text{ be corresponding EDM .}\)

\((to \ D \in \mathcal{E}^n)\)

\[
D = \left( \|p_i - p_j\|_2^2 \right)_{i,j=1}^n \\
= \left( p_i^T p_i + p_j^T p_j - 2 p_i^T p_j \right)_{i,j=1}^n \\
= \text{diag } (B) e^T + e \text{ diag } (B)^T - 2B \\
=: D_e(B) - 2B \\
=: \mathcal{K}(B) \quad (\text{from } B \in S^n_+).
\]
Current Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax \( \text{rank } B \))

- \( \min_{B \succeq 0, B \in \Omega} \| H \circ (\mathcal{K}(B) - D) \| ; \text{rank } B = r \);
- typical weights: \( H_{ij} = 1 / \sqrt{D_{ij}} \), if \( ij \in E \).
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, **BUT**:
  - expensive
  - low accuracy
  - implicitly highly degenerate (cliques restrict ranks of feasible \( Bs \))
Instead: Take Advantage of Implicit Degeneracy!

- clique $\alpha, |\alpha| = k$ given
- (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$
  \[ \implies \text{rank } K^\dagger(D[\alpha]) = t \leq r \]
  \[ \implies \text{rank } B[\alpha] \leq \text{rank } K^\dagger(D[\alpha]) + 1 \]
  \[ \text{rank } B = \text{rank } K^\dagger(D) \leq n - (k - t - 1) \]
  \[ \implies \]
  Slater’s CQ (strict feasibility) fails
  a proper face containing feasible set of $B$s can be identified.
(\mathcal{S}^n : ) \quad \mathcal{K} : \mathcal{S}_+^n \cap \mathcal{S}_C \rightarrow \mathcal{E}^n \subset \mathcal{S}^n \cap \mathcal{S}_H \quad \leftarrow : \mathcal{T} \quad ( : \mathcal{E}^n )

Linear Transformations: \( \mathcal{D}_v(B), \mathcal{K}(B), \mathcal{T}(D) \)

- allow: \( \mathcal{D}_v(B) := \text{diag}(B) v^T + v \text{diag}(B)^T \);\( \mathcal{D}_v(y) := yv^T + vy^T \)
- adjoint \( \mathcal{K}^*(D) = 2(\text{Diag}(De) - D) \).
- \( \mathcal{K} \) is \( 1-1 \), onto between centered & hollow subspaces:
  - \( \mathcal{S}_C := \{ B \in \mathcal{S}^n : Be = 0 \} \);
  - \( \mathcal{S}_H := \{ D \in \mathcal{S}^n : \text{diag}(D) = 0 \} = \mathcal{R}(\text{offDiag}) \)
- \( J := I - \frac{1}{n}ee^T \) (orthogonal projection onto \( \mathcal{M} := \{ e \}^\perp \));
- \( \mathcal{T}(D) := -\frac{1}{2}J \text{offDiag}(D)J \quad (= \mathcal{K}^\dagger(D)) \)
Properties of Linear Transformations

\[ \mathcal{K}, \mathcal{T}, \text{Diag}, \mathcal{D}_e \]

\[ \mathcal{R}(\mathcal{K}) = S_H; \quad \mathcal{N}(\mathcal{K}) = \mathcal{R}(\mathcal{D}_e); \]
\[ \mathcal{R}(\mathcal{K}^*) = \mathcal{R}(\mathcal{T}) = S_C; \quad \mathcal{N}(\mathcal{K}^*) = \mathcal{N}(\mathcal{T}) = \mathcal{R}(\text{Diag}); \]

\[ S^n = S_H \oplus \mathcal{R}(\text{Diag}) = S_C \oplus \mathcal{R}(\mathcal{D}_e). \]

\[ \mathcal{T}(\mathcal{E}^n) = S^n_+ \cap S_C \quad \text{and} \quad \mathcal{K}(S^n_+ \cap S_C) = \mathcal{E}^n. \]
Semidefinite Cone, Faces

- \( F \subseteq K \) is a **face of** \( K \), denoted \( F \lessdot K \), if
  \[
  (x, y \in K, \frac{1}{2}(x + y) \in F) \implies \text{cone}\{x, y\} \subseteq F.
  \]
- All faces of \( S^n_+ \) are exposed.

**Faces of cone \( K \)**

- \( F \lessdot K \), if \( F \lessdot K, F \neq K; \) \( F \) is **proper face** if \( \{0\} \neq F \lessdot K \).
- \( F \lessdot K \) is **exposed** if: intersection of \( K \) with a hyperplane.
- **face**\((S)\) denotes smallest face of \( K \) that contains set \( S \).
Facial Structure of SDP Cone; Equivalent SUBSPACES

**Face** $F \leq S^n_+$ Equivalence to $\mathcal{R}(U)$ Subspace of $\mathbb{R}^n$

$F \leq S^n_+$ determined by range of any $S \in \text{relint } F$,
i.e. let $S = U\Gamma U^T$ be compact spectral decomposition; $\Gamma \in S^{++}_t$
is diagonal matrix of pos. eigenvalues;

\[
F = US^t_+ U^T
\]

($F$ associated with $\mathcal{R}(U)$)

\[
\text{dim } F = t(t+1)/2.
\]

Face $F$ represented by subspace $\mathcal{L}$ or matrix $T$
(subspace) $\mathcal{L} = \mathcal{R}(T)$, $T$ is $n \times t$ full column, then:

\[
F := TS^t_+ T^T \leq S^n_+
\]
Facial Structure of SDP Cone; Equivalent SUBSPACES

Face $F \trianglelefteq S^n_+$ Equivalence to $\mathcal{R}(U)$ Subspace of $\mathbb{R}^n$

$F \trianglelefteq S^n_+$ determined by range of any $S \in \text{relint } F$, i.e. let $S = U\Gamma U^T$ be compact spectral decomposition; $\Gamma \in S^{t+}_{++}$ is diagonal matrix of pos. eigenvalues; $F = US_t^+U^T$

($F$ associated with $\mathcal{R}(U)$)

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$F := TS_t^+T^T \trianglelefteq S^n_+$
Further Notation

Matrix with Fixed Principal Submatrix

For $Y \in S^n$, $\alpha \subseteq \{1, \ldots, n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices $\alpha$.

Sets with Fixed Principal Submatrices

If $|\alpha| = k$ and $\bar{Y} \in S^k$, then:

- $S^n(\alpha, \bar{Y}) := \{Y \in S^n : Y[\alpha] = \bar{Y}\}$,
- $S^n_+(\alpha, \bar{Y}) := \{Y \in S^n_+ : Y[\alpha] = \bar{Y}\}$

i.e. the subset of matrices $Y \in S^n$ ($Y \in S^n_+$) with principal submatrix $Y[\alpha]$ fixed to $\bar{Y}$. 
Further Notation

**Matrix with Fixed Principal Submatrix**

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**Sets with Fixed Principal Submatrices**

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- $S^+_n(\alpha, \bar{Y}) := \{ Y \in S^+_n : Y[\alpha] = \bar{Y} \}$

i.e. the subset of matrices $Y \in S^n$ ($Y \in S^+_n$) with principal submatrix $Y[\alpha]$ fixed to $\bar{Y}$. 
Basic Single Clique/Facial Reduction

\[ \bar{D} \in E^k, \alpha \subseteq 1:n, |\alpha| = k \]

Define \[ E^n(\alpha, \bar{D}) := \{ D \in E^n : D[\alpha] = \bar{D} \}. \]

Given \( \bar{D} \); find a corresponding \( B \succeq 0 \); find the corresponding face; find the corresponding subspace.

if \( \alpha = 1:k \); embed. dim of \( \bar{D} \) is \( t \leq r \)

\[
D = \begin{bmatrix} \bar{D} \\ \vdots \end{bmatrix},
\]
Basic Single Clique/Facial Reduction

\[ \bar{D} \in \mathcal{E}^k, \alpha \subseteq 1:n, |\alpha| = k \]

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\[ D = \begin{bmatrix} \bar{D} \\ . \\ . \end{bmatrix}, \]
BASIC THEOREM 1: Single Clique/Facial Reduction

Let: \( \tilde{D} := D[1:k] \in \mathcal{E}^k, \ k < n \), with embedding dimension \( t \leq r \); 
\( B := \mathcal{K}^\dagger(\tilde{D}) = \bar{U}_B S \bar{U}_B^T \), \( \bar{U}_B \in \mathcal{M}^{k \times t} \), \( \bar{U}_B^T \bar{U}_B = I_t \), \( S \in S_t^{++} \).

Furthermore, let \( \bar{U}_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k \times (t+1)} \), 
\( U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix} \), and let \( V \begin{bmatrix} U^T e \\ \| U^T e \| \end{bmatrix} \in \mathcal{M}^{n-k+t+1} \) be orthogonal.

Then:

\[
\text{face } \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \tilde{D})) = (US^{n-k+t+1}_+ U^T) \cap S_C
= (UV)S^{n-k+t}_+ (UV)^T
\]
**BASIC THEOREM 1:** Single Clique/Facial Reduction

Let: \( \bar{D} := D[1:k] \in \mathcal{E}^k, k < n \), with embedding dimension \( t \leq r \);
\( B := \mathcal{K}^\dagger (\bar{D}) = \bar{U}_B S \bar{U}_B^T, \bar{U}_B \in \mathcal{M}^{k \times t}, \bar{U}_B^T \bar{U}_B = I_t, S \in S_t^{++} \).

Furthermore, let \( U_B := \left[ \begin{array}{cc} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{array} \right] \in \mathcal{M}^{k \times (t+1)} \),
\( U := \left[ \begin{array}{cc} U_B & 0 \\ 0 & I_{n-k} \end{array} \right] \), and let \( V \left( \begin{array}{c} U^T e \\ \Vert U^T e \Vert \end{array} \right) \in \mathcal{M}^{n-k+t+1} \) be orthogonal.

Then:

\[
\text{face } \mathcal{K}^\dagger (\mathcal{E}^n(1:k, \bar{D})) = \left( U S_{++}^{n-k+t+1} U^T \right) \cap S_c \\
= (U V) S_{++}^{n-k+t} (U V)^T
\]
BASIC THEOREM 1: Single Clique/Facial Reduction

Let: \( \bar{D} := D[1:k] \in \mathcal{E}^k, k < n \), with embedding dimension \( t \leq r \); 
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Then:

\[
\text{face } K^\dagger(\mathcal{E}^n(1:k, \bar{D})) = (US_n^{n-k+t+1} U^T) \cap S_C = (UV)S_n^{n-k+t} (UV)^T
\]
Sets for Intersecting Clique/Faces

\[\alpha_1 := 1 : (\bar{k}_1 + \bar{k}_2); \quad \alpha_2 := (\bar{k}_1 + 1) : (\bar{k}_1 + \bar{k}_2 + \bar{k}_3)\]

For each clique \(|\alpha| = k\), we get a corresponding face/subspace \((k \times r)\) representation. We now see how to handle two cliques, \(\alpha_1, \alpha_2\), that intersect.

\[ \alpha_1, \alpha_2 \subseteq 1:n; \quad k := |\alpha_1 \cup \alpha_2| \]

For \( i = 1, 2 \): \( \bar{D}_i := D[\alpha_i] \in \mathbb{E}^{k_i} \), embedding dimension \( t_i \);

\[ B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^T, \quad \bar{U}_i \in \mathcal{M}^{k_i \times t_i}, \quad \bar{U}_i^T \bar{U}_i = I_{t_i}, \quad S_i \in S^{t_i}_{++}; \]

\[ U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}; \quad \text{and} \quad \bar{U} \in \mathcal{M}^{k \times (t+1)} \]

satisfies

\[ \mathcal{R}(\bar{U}) = \mathcal{R} \left( \begin{bmatrix} U_1 & 0 \end{bmatrix} \right) \cap \mathcal{R} \left( \begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix} \right), \quad \text{with} \quad \bar{U}^T \bar{U} = I_{t+1} \]

(intersection of subspaces)

cont...

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\( U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}; \) and \( \bar{U} \in \mathcal{M}^{k \times (t+1)} \) satisfies

\[ \mathcal{R}(\bar{U}) = \mathcal{R} \left( \begin{bmatrix} U_1 & 0 \\ 0 & I_{k_3} \end{bmatrix} \right) \cap \mathcal{R} \left( \begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix} \right), \]

(with \( \bar{U}^T \bar{U} = I_{t+1} \))

(intersection of subspaces)

cont. . .
Two (Intersecting) Clique Reduction, cont.

**THEOREM 2** Nonsing. Clique/Facial Inters. cont.

\[ \mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & L_{k_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix}\right) \text{, with } \bar{U}^T\bar{U} = I_{t+1}; \]

let: 
\[ U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M} \]

and 
\[ \begin{bmatrix} V \\ \frac{U^Te}{\|U^Te\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1} \]

be orthogonal. Then

\[ \bigcap_{i=1}^2 \text{face } \mathcal{K}^\dagger (\mathcal{E}^n(\alpha_i, \bar{D}_i)) = \left(US_{n-k+t+1}^+U^T\right) \cap S_C = (UV)S_{n-k+t}^+(UV)^T \]
Expense/Work of (Two) Clique/Facial Reductions

Subspace Intersection for Two Intersecting Cliques/Faces

Suppose:

\[
U_1 = \begin{bmatrix} U'_1 & 0 \\ U''_1 & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U''_2 \\ 0 & U'_2 \end{bmatrix}
\]

Then:

\[
U := \begin{bmatrix} U'_1 & U''_1 \\ U''_1 & U'_1 (U''_1)^\dagger U''_1 \\ U'_2 (U''_2)^\dagger U''_1 \\ U''_2 \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U'_1 (U''_1)^\dagger U''_2 \\ U''_2 \\ U'_2 \end{bmatrix}
\]

(Efficiently/accurately) satisfies:

\[
\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)
\]
Two (Intersecting) Clique Reduction Figure

Completion: missing distances can be recovered if desired.
COR: (Intersect.) Clique Explicit \textit{Delayed} Completion

Hypotheses of Theorem 2 holds; \( \bar{D}_i := D[\alpha_i] \in \mathcal{E}^k_i \), for \( i = 1, 2 \), \( \beta \subseteq \alpha_1 \cap \alpha_2 \), \( \gamma := \alpha_1 \cup \alpha_2 \), \( \bar{D} := D[\beta] \)

\[
B := \mathcal{K}^\dagger(\bar{D}), \quad \bar{U}_\beta := \bar{U}(\beta, :), \text{ where } \bar{U} \in \mathcal{M}^{k \times (t+1)} \text{ satisfies intersection equation of Theorem 2. Let } \begin{bmatrix} \bar{V} & \bar{U}^T e \end{bmatrix} \in \mathcal{M}^{t+1} \text{ be orthogonal. Let } Z := (J\bar{U}_\beta \bar{V})^\dagger B((J\bar{U}_\beta \bar{V})^\dagger)^T. 
\]

If the embedding dimension for \( \bar{D} \) is \( r \), \textbf{THEN} \( t = r \) in Theorem 2, and \( Z \in \mathcal{S}_+^r \) is the unique solution of the equation \((J\bar{U}_\beta \bar{V})Z(J\bar{U}_\beta \bar{V})^T = B \), and the exact completion is \[
D[\gamma] = \mathcal{K} \left( PP^T \right) \quad \text{where } P := UVZ_\frac{1}{2} \in \mathbb{R}^{|\gamma| \times r}
\]
Use $R$ as lower bound in singular/nonrigid case.
Two (Inters.) Clique Explicit Compl.; Sing. Case

COR. Clique-Sing.; Intersect. Embedding Dim. $r - 1$

Hypotheses of previous COR holds. For $i = 1, 2$, let $\beta \subset \delta_i \subseteq \alpha_i$, $A_i := J\bar{U}_\delta V$, where $\bar{U}_\delta := \bar{U}(\delta_i, :)$, and $B_i := \mathcal{K}^\dagger(D[\delta_i])$. Let $\bar{Z} \in S^t$ be a particular solution of the linear systems

\[
\begin{align*}
A_1 Z A_1^T &= B_1 \\
A_2 Z A_2^T &= B_2.
\end{align*}
\]

If the embedding dimension of $D[\delta_i]$ is $r$, for $i = 1, 2$, but the embedding dimension of $\bar{D} := D[\beta]$ is $r - 1$, then the following holds. cont...
2 (Inters.) Clique Expl. Compl.; Degen. cont...
Completing SNL (Delayed use of Anchor Locations)

Rotate to Align the Anchor Positions

- Given \( P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r} \) such that \( D = \mathcal{K}(PP^T) \)

- Solve the orthogonal Procrustes problem:

\[
\begin{align*}
\min & \quad \| A - P_2 Q \| \\
\text{s.t.} & \quad Q^T Q = I
\end{align*}
\]

\[
P_2^T A = U\Sigma V^T \quad \text{SVD decomposition; set } Q = UV^T; \quad \text{(Golub/Van Loan, Algorithm 12.4.1)}
\]

- Set \( X := P_1 Q \)
Algorithm: Four Cases

<table>
<thead>
<tr>
<th></th>
<th>Clique Union</th>
<th>Node Absorption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid</td>
<td><img src="image1" alt="Clique Union Diagram" /></td>
<td><img src="image2" alt="Node Absorption Diagram" /></td>
</tr>
<tr>
<td>Non-rigid</td>
<td><img src="image3" alt="Clique Union Diagram" /></td>
<td><img src="image4" alt="Node Absorption Diagram" /></td>
</tr>
</tbody>
</table>

Preliminaries
Clique/Facial Reduction (Exploit degeneracy)
Algorithm: Facial Reduct. via Subsp. Inters./DELAYED Compl.
Noisy Data
Summary
Clique Unions and Node Absorptions
Results (low CPU time; high accuracy)
ALGOR: clique union; facial reduct.; delay compl.

Initialize: Find initial set of cliques.

\[ C_i := \{ j : (D_p)_{ij} < (R/2)^2 \} \quad \text{for} \quad i = 1, \ldots, n \]

Iterate

- For \( |C_i \cap C_j| \geq r + 1 \), do Rigid Clique Union
- For \( |C_i \cap N(j)| \geq r + 1 \), do Rigid Node Absorption
- For \( |C_i \cap C_j| = r \), do Non-Rigid Clique Union (lower bnds)
- For \( |C_i \cap N(j)| = r \), do Non-Rigid Node Absorp. (lower bnds)

Finalize

When \( \exists \) a clique containing all anchors, use computed facial representation and positions of anchors to solve for \( X \)
**ALGOR:** clique union; facial reduct.; delay compl.

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Finalize

When \(\exists\) a clique containing all anchors, use computed facial representation and positions of anchors to solve for \(X\)
Results - Data for Random Noisless Problems

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension \( r = 2 \)
- Square region: \([0, 1] \times [0, 1]\)
- \( m = 9 \) anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

\[
\text{RMSD} = \left( \frac{1}{n} \sum_{i=1}^{n} \| p_i - p_i^{\text{true}} \|^2 \right)^{1/2}
\]
**Results - Large \( n \) (SDP size \( O(n^2) \))**

<table>
<thead>
<tr>
<th>( n ) # sensors ( \setminus R )</th>
<th>0.07</th>
<th>0.06</th>
<th>0.05</th>
<th>0.04</th>
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<td>6000</td>
<td>6000</td>
<td>6000</td>
<td>6000</td>
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<td>10000</td>
<td>10000</td>
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</table>

<table>
<thead>
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<th>0.06</th>
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<th>0.04</th>
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<td>1</td>
<td>1</td>
<td>3</td>
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<tr>
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<tr>
<td>10000</td>
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<td>10</td>
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<table>
<thead>
<tr>
<th>( n ) # sensors ( \setminus R )</th>
<th>0.07</th>
<th>0.06</th>
<th>0.05</th>
<th>0.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>(3e^{-16})</td>
<td>(5e^{-16})</td>
<td>(6e^{-16})</td>
<td>(3e^{-16})</td>
</tr>
<tr>
<td>6000</td>
<td>(3e^{-16})</td>
<td>(4e^{-16})</td>
<td>(3e^{-16})</td>
<td>(3e^{-16})</td>
</tr>
<tr>
<td>10000</td>
<td>(3e^{-16})</td>
<td>(5e^{-16})</td>
<td>(4e^{-16})</td>
<td>(4e^{-16})</td>
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</tbody>
</table>
Results - *N* Huge SDPs Solved

### Large-Scale Problems

<table>
<thead>
<tr>
<th># sensors</th>
<th># anchors</th>
<th>radio range</th>
<th>RMSD</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>20000</td>
<td>9</td>
<td>.025</td>
<td>5e−16</td>
<td>25s</td>
</tr>
<tr>
<td>40000</td>
<td>9</td>
<td>.02</td>
<td>8e−16</td>
<td>1m 23s</td>
</tr>
<tr>
<td>60000</td>
<td>9</td>
<td>.015</td>
<td>5e−16</td>
<td>3m 13s</td>
</tr>
<tr>
<td>100000</td>
<td>9</td>
<td>.01</td>
<td>6e−16</td>
<td>9m 8s</td>
</tr>
</tbody>
</table>

Size of SDPs Solved: \( N = \binom{n}{2} \) (# vrbls)

\[ E(\text{density of } G) = \pi R^2; \quad M = E(|E|) = \pi R^2 N \] (# constraints)

Size of SDP Problems:

\[
M = [3,078,915, 12,315,351, 27,709,309, 76,969,790] \\
N = 10^9 [0.2000, 0.8000, 1.8000, 5.0000] 
\]
Locally Recover Exact EDMs

**Nearest EDM**

- Given clique $\alpha$; corresp. EDM $D_\epsilon = D + N_\epsilon$, $N_\epsilon$ noise
- we need to find the smallest face containing $E^n(\alpha, D)$.

$$\begin{align*}
\min & \quad \|K(X) - D_\epsilon\| \\
\text{s.t.} & \quad \text{rank}(X) = r, Xe = 0, X \succeq 0 \\
& \quad X \succeq 0.
\end{align*}$$

- Eliminate the constraints: $Ve = 0$, $V^TV = I$, $K_V(X) := K(VXV^T)$:

$$U^*_r \in \arg\min \frac{1}{2} \|K_V(UU^T) - D_\epsilon\|_F^2 \quad \text{s.t.} \quad U \in M^{(n-1)r}.$$

The nearest EDM is $D^* = K_V(U^*_r(U^*_r)^T)$.
Solve Overdetermined Nonlin. Least Squares Prob.

Newton (expensive) or Gauss-Newton (less accurate)

\[ F(U) := \text{us2vec} \left( K \ V(UU^T) - D_\epsilon \right), \quad \min_U f(U) := \frac{1}{2} \| F(U) \|^2 \]

Derivatives: gradient and Hessian

\[ \nabla f(U)(\Delta U) = \langle 2 \left( K^* V \left[ K \ V(UU^T) - D_\epsilon \right] \right) U, \Delta U \rangle \]

\[ \nabla^2 f(U) = 2 \text{vec} \left( L^*_U K^* K \ V S_{\Sigma} L \ U + K^*_V \left( K \ V(UU^T) - D_\epsilon \right) \right) \text{Mat} \]

where \( L_U(\cdot) = \cdot U^T \); \( S_{\Sigma}(U) = \frac{1}{2}(U + U^T) \)
random noisy probs; \( r = 2, m = 9, nf = 1 \times 10^{-6} \)

## Using only Rigid Clique Union, preliminary results:

### remaining cliques

<table>
<thead>
<tr>
<th>n / R</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
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<tr>
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<td>5.00</td>
<td>11.00</td>
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<td>1.00</td>
<td>1.00</td>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
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<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### cpu seconds

<table>
<thead>
<tr>
<th>n / R</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
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<td>4.05</td>
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<tr>
<td>2000</td>
<td>12.46</td>
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<td>12.43</td>
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<td>24.01</td>
<td>24.02</td>
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<tr>
<td>5000</td>
<td>38.13</td>
<td>31.66</td>
<td>30.26</td>
<td>30.32</td>
<td>29.88</td>
</tr>
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</table>

### max-log-error

<table>
<thead>
<tr>
<th>n / R</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-2.92</td>
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<td>Inf</td>
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<tr>
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<tr>
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<td>-3.98</td>
<td>-3.25</td>
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<td>-3.52</td>
<td>-3.04</td>
<td>-3.33</td>
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<tr>
<td>5000</td>
<td>-4.80</td>
<td>-4.38</td>
<td>-3.89</td>
<td>-4.13</td>
<td>-3.40</td>
</tr>
</tbody>
</table>
Summary

- SDP relaxation of SNL is (highly, implicitly) degenerate: feasible set is restricted to a low dim. face (Slater CQ - strict feasibility - fails)
- take advantage of degeneracy using explicit representations of intersections of faces corresponding to unions of intersecting cliques
- Without using an SDP-solver, we efficiently compute exact solutions to SDP relaxation (dual/extended view of geometric buildup)
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Thanks for your attention!

Explicit Sensor Network Localization using Semidefinite Programming and Facial Reduction

Nathan Krislock and Henry Wolkowicz

Dept. of Combinatorics and Optimization
University of Waterloo

at ICME, Stanford University
Friday, Oct. 30, 2009