

MOTZKIN'S TRANSPOSITION THEOREM, AND THE RELATED THEOREMS OF FARKAS, GORDAN AND STIEMKE

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Motzkin's thesis [6], in particular his *Transposition Theorem* (Theorems 1–2 below), was a milestone in the development of linear inequalities and related areas.

For two vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_i)$ of equal dimension we denote by $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} > \mathbf{v}$ that the indicated inequality holds componentwise, and by $\mathbf{u} \not\geq \mathbf{v}$ the fact $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.

Systems of linear inequalities appear in several forms; the following examples are typical:

- (a) $A\mathbf{x} \leq \mathbf{b}$ (b) $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ (c) $A\mathbf{x} \leq \mathbf{b}$, $B\mathbf{x} < \mathbf{c}$
 (d) $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ (e) $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ (f) $A\mathbf{x} > \mathbf{0}$, $B\mathbf{x} \geq \mathbf{0}$, $C\mathbf{x} = \mathbf{0}$

In each of these systems, called *primal*, the existence of solutions is characterized by means of a *dual system*, using the transposes of matrices in the primal system. Hence the name *transposition theorem*. The relation between the primal and dual systems is sometimes given as a *theorem of alternatives*, listing *alternatives*, i.e. statements P, Q satisfying $P \iff \neg Q$ (\neg denotes negation), in words: *either P or Q but never both*.

Relations between the above systems: (a) and (b) are equivalent representations. Indeed, (a) and (b) can be written as

$$(A, -A, I) \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \end{pmatrix} = \mathbf{b}, \quad \begin{pmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \end{pmatrix} \geq \mathbf{0} \quad \text{and} \quad \begin{pmatrix} A \\ -A \\ -I \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{pmatrix}, \quad \text{respectively.}$$

The remaining systems involve strict inequalities or nontrivial solutions. For example, (d) and (e) concern the existence of nontrivial solutions and positive solutions, respectively, for the positively homogeneous system

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}.$$

Taking $B = O$ and $\mathbf{c} > \mathbf{0}$ in (c) gives (a). Therefore, (a) and (b) are special cases of (c). Similarly, the systems (d) and (e) are special cases of (f), which itself is a special case of (c) with $\mathbf{b} = \mathbf{0}$, $\mathbf{c} = \mathbf{0}$. In fact, every system of linear inequalities can be written as (c).

The following two versions of *Motzkin's Transposition Theorem*, [6], concern systems (c) and (f).

Theorem 1 (Solvability of (c)). Given matrices A, B and vectors \mathbf{b}, \mathbf{c} , the following are equivalent:

- (c1) the system $A\mathbf{x} \leq \mathbf{b}$, $B\mathbf{x} < \mathbf{c}$ has a solution \mathbf{x}
 (c2) for all vectors $\mathbf{y} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$,
 $A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{0} \implies \mathbf{b}^T\mathbf{y} + \mathbf{c}^T\mathbf{z} \geq 0$ and
 $A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{0}$, $\mathbf{z} \neq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} + \mathbf{c}^T\mathbf{z} > 0$. □

Theorem 2 (Solvability of (f)). Let A, B, C be given matrices, with A nonvacuous. Then the following are alternatives:

- (f1) $A\mathbf{x} > \mathbf{0}$, $B\mathbf{x} \geq \mathbf{0}$, $C\mathbf{x} = \mathbf{0}$ has a solution \mathbf{x} ,
 (f2) $A^T\mathbf{y}_1 + B^T\mathbf{y}_2 + C^T\mathbf{y}_3 = \mathbf{0}$, $\mathbf{y}_1 \not\geq \mathbf{0}$, $\mathbf{y}_2 \geq \mathbf{0}$ has solutions $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$, □

Special cases of Motzkin's Theorem include the following four theorems. First, the celebrated Farkas' Theorem, [2].

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Theorem 3 (Farkas' Theorem for system (a)). Given a matrix A and a vector \mathbf{b} , the following are equivalent:

(a1) the system $A\mathbf{x} \leq \mathbf{b}$ has a solution \mathbf{x}

(a2) $A^T\mathbf{y} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} \geq 0$. □

Theorem 4 (Farkas' Theorem for system (b)). Given a matrix A and a vector \mathbf{b} , the following are equivalent:

(b1) the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a solution \mathbf{x}

(b2) $A^T\mathbf{y} \geq \mathbf{0} \implies \mathbf{b}^T\mathbf{y} \geq 0$. □

The positively homogeneous systems (d) and (e) are covered by the following two theorems.

Theorem 5 ([3]). Given a matrix A , the following are alternatives:

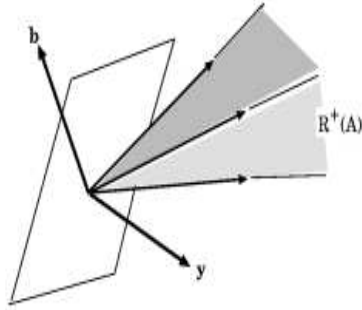
(d1) $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \not\geq \mathbf{0}$ has a solution \mathbf{x} ,

(d2) $A^T\mathbf{y} > \mathbf{0}$ has a solution \mathbf{y} □

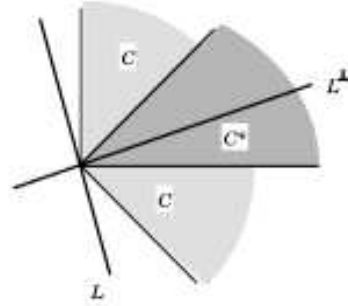
Theorem 6 ([10]). Given a matrix A , the following are alternatives:

(e1) $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ has a solution \mathbf{x} ,

(e2) $A^T\mathbf{y} \not\geq \mathbf{0}$ has a solution \mathbf{y} , □



(a) A hyperplane with normal \mathbf{y} separating \mathbf{b} and $\mathbb{R}^+(A)$



(b) Illustration of the alternatives (6)
 $L \cap C = \{\mathbf{0}\}$, $L^\perp \cap \text{int } C^* \neq \emptyset$

FIGURE 1. Illustrations

The above results are *Separation Theorems*, or statements about the existence of hyperplanes separating certain disjoint convex sets. First, some terminology. A set $P \subset \mathbb{R}^n$ is *polyhedral* (and necessarily convex) if it is the intersection of finitely many closed halfspaces, say

$$P := \{\mathbf{x} : B\mathbf{x} \leq \mathbf{b}\}, \text{ for some matrix } B \text{ and vector } \mathbf{b}. \quad (1)$$

A *finitely generated cone* is the set of nonnegative linear combinations of finitely many vectors (*generators*), for example, the cone generated by the columns of a matrix A ,

$$\mathbb{R}^+(A) := \{A\mathbf{x} : \mathbf{x} \geq \mathbf{0}\}. \quad (2)$$

The *dual* (or *polar*) S^* of a nonempty set $S \subset \mathbb{R}^n$ is defined as

$$S^* := \{\mathbf{y} : \mathbf{s} \in S \implies \mathbf{y}^T\mathbf{s} \geq 0\}, \quad (3)$$

a closed convex cone. In particular,

$$(\mathbb{R}^+(A))^* = \{\mathbf{y} : A^T\mathbf{y} \geq \mathbf{0}\}, \text{ a polyhedral cone}. \quad (4)$$

Theorem 4(b1) states that the vector \mathbf{b} is in the cone $\mathbb{R}^+(A)$. The equivalent statement (b2) says that \mathbf{b} cannot be separated from $\mathbb{R}^+(A)$ by a hyperplane: such a separating hyperplane would have a normal \mathbf{y} , see Fig. 1(a), satisfying

$$\mathbf{b}^T\mathbf{y} < 0 \text{ and } \mathbf{v}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{v} \in \mathbb{R}^+(A),$$

which by (4) is a negation of (b2). Farkas' Theorem 4 states that for any matrix A ,

$$\mathbb{R}^+(A) = (\mathbb{R}^+(A))^{**} . \quad (5)$$

In general, a set $C \subset \mathbb{R}^n$ is a closed convex cone if and only if $C = C^{**}$. Theorem 4 also implies that a cone in \mathbb{R}^n is polyhedral if and only if it is finitely generated (the *Farkas-Minkowski-Weyl Theorem*, [9, Corollary 7.1a]. More generally, a set $S \subset \mathbb{R}^n$ is polyhedral if and only if it is the sum of a finitely generated cone and the convex hull of finitely many points (the *Minkowski-Steinitz-Weyl Theorem*, [9, Corollary 7.1b].

Theorems 5–6 can be interpreted as geometric statements about intersections $C \cap L$, of a closed convex cone C and a subspace L in \mathbb{R}^n . Let \mathbb{R}_+^n denote the nonnegative orthant in \mathbb{R}^n . Thus:

Theorem 5(d1): $\mathbb{R}_+^n \cap N(A) \neq \mathbf{0}$, where $N(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ the *nullspace* of A .

Theorem 6(e1): $\text{int}(\mathbb{R}_+^n) \cap N(A) \neq \emptyset$, where $\text{int}(\mathbb{R}_+^n) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} > \mathbf{0}\}$.

In each case, the dual system uses the intersection $C^* \cap L^\perp$ where L^\perp is the *orthogonal complement* of L . For example, the statements (\exists denotes *there exists*, *int* denotes *interior*)

$$\exists \mathbf{0} \neq \mathbf{x} \in C \cap L \quad \text{and} \quad \exists \mathbf{y} \in (\text{int } C^*) \cap L^\perp \quad (6)$$

are mutually exclusive, see Fig. 1(b), for otherwise

$$\mathbf{x}^T \mathbf{y} \begin{cases} = 0 & \text{since } \mathbf{x} \perp \mathbf{y} \\ > 0 & \text{since } \mathbf{0} \neq \mathbf{x} \in C, \mathbf{y} \in \text{int } C^* \end{cases}$$

To make the statements in (6) alternatives, we need to show that one of them occurs, the hard part of the proof. Returning to Theorems 5–6, recall that $(\mathbb{R}_+^n)^* = \mathbb{R}_+^n$ and $N(A)^\perp = R(A^T)$. Then:

Gordan's Theorem: (d1) $\mathbb{R}_+^n \cap N(A) \neq \mathbf{0}$ and (d2) $\text{int}(\mathbb{R}_+^n) \cap R(A^T)$ are alternatives.

Stiemke's Theorem: (e1) $\text{int}(\mathbb{R}_+^n) \cap N(A) \neq \emptyset$ and (e2) $\mathbb{R}_+^n \cap R(A^T) \neq \mathbf{0}$ are alternatives.

Further readings:

History: [9, pp. 209–228].

Theorems of alternatives: [5, pp. 27–37].

Generalizations: [11],[1],[8, §§ 21–22, specially Theorems 21.1,22.6].

Applications: [7],[5, p. 100].

Biographical note: Theodore S. Motzkin (Basel, 1900–Los Angeles, 1970) made important contributions to linear inequalities and polyhedral combinatorics. His name is associated with the *Motzkin transposition theorem*, *Motzkin numbers*, *Motzkin paths*, *Fourier-Motzkin elimination method* and its dual, the *double description method*. His father, Leo Motzkin (1867–1933), who studied Mathematics and Sociology at the University of Berlin, was a Zionist politician and cultural leader. The Israeli city of Kiryat Motzkin is named after him.

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