Optimization problems in compressed sensing

Jalal Fadili

CNRS, ENSI Caen France

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Today’s talk is about ...

- Compressed sensing.
- Sparse representations.
- Convex analysis and operator splitting.
  - Non-smooth optimization.
  - Monotone operator splitting.
  - Fast algorithms.
Compressed/ive Sensing
Compressed/ive Sensing

- Common wisdom: Shannon sampling theorem and bandlimited signals.
Compressed/ive Sensing

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- The CS theory asserts that one can recover certain (sparse) signals and images from far fewer measurements $m$ than data samples $n$.
Compressed/ive Sensing

- Common wisdom: Shannon sampling theorem and bandlimited signals.

- The CS theory asserts that one can recover certain (sparse) signals and images from far fewer measurements $m$ than data samples $n$.

- The CS acquisition scenario: $\mathbf{y} = \mathbf{H}\mathbf{x} = (\langle \mathbf{x}, \mathbf{h}_i \rangle)_{i=1}^m, \; m \ll n$. 

\[
\begin{align*}
\mathbf{y} & \quad m \times 1 \\
\mathbf{H} & \quad m \times n \\
\Phi & \quad n \times L \\
\mathbf{a} & \quad L \times 1 \\
\mathbf{x} & \quad n \times 1
\end{align*}
\]
Common wisdom: Shannon sampling theorem and bandlimited signals.

The CS theory asserts that one can recover certain (sparse) signals and images from far fewer measurements $m$ than data samples $n$.

The CS acquisition scenario: $\mathbf{y} = \mathbf{Hx} = (\langle \mathbf{x}, \mathbf{h}_i \rangle)_{i=1}^m$, $m \ll n$. 
Compressed/sive Sensing (cont’d)
CS relies on two tenets:
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- **Sparsity (compressibility):** $x$ is sparse in $\Phi$.
- **Incoherence:** the sensing vectors $\left( h_i \right)_{i=1}^{m}$ as different as possible from the sparsity waveforms $\left( \varphi_j \right)_{j=1}^{L}$. 
Compressed/ive Sensing (cont’d)

CS relies on two tenets:

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- **Incoherence**: the sensing vectors $(\mathbf{h}_i)_{i=1}^m$ as different as possible from the sparsity waveforms $(\varphi_j)_{j=1}^L$.

CS decoder $\Delta_\Phi(y) : \mathbb{R}^m \rightarrow \mathbb{R}^L$ proposes to recover the signal/image by solving the non-linear program

$$(P_{eq}) : \min \|\alpha\|_{\ell_1} \quad \text{s.t.} \quad y = H\Phi\alpha.$$
Construct $H$ : Random Sensing

Random Sensing: $\alpha$ is $s$-sparse and given $m$ measurements selected uniformly at random from an ensemble. If $m \geq C's\mu_H^2\log n$, then minimizing $(P_{eq})$ reconstructs $\alpha$ exactly with overwhelming probability.

Compressible signals/images and $\ell_2 - \ell_1$ instance optimality: $(P_{eq})$ solution recovers the $s$-largest entries.

Stability to noise $y = H\Phi\alpha + \varepsilon$: the decoder $\Delta_{\Phi,\sigma}(y)$

$$(P_\sigma) : \min \|\alpha\|_{\ell_1} \text{ s.t. } \|y - H\Phi\alpha\|_{\ell_2} \leq \sigma$$

has $\ell_2 - \ell_1$ instance optimality and a factor of the noise std $\sigma$. 
Convex analysis and operator splitting
Class of problems in CS

- Recall $y = Hx + \varepsilon$, $\varepsilon$ iid and $\text{Var}(\varepsilon) = \sigma^2 < +\infty$,

  - $x = \Phi \alpha$, $\alpha$ (nearly-)sparse.

- Typical (equivalent) minimization problems:

  $$(P_{\text{eq}}) : \quad \min \Psi(\alpha) \quad \text{s.t.} \quad y = H\Phi \alpha$$

  $$(P_{\sigma}) : \quad \min \Psi(\alpha) \quad \text{s.t.} \quad \|y - H\Phi \alpha\|_2 \leq \sigma$$

  $$(P_{\tau}) : \quad \min \frac{1}{2} \|y - H\Phi \alpha\|_2^2 \quad \text{s.t.} \quad \Psi(\alpha) \leq \tau$$

  $$(P_{\lambda}) : \quad \min \frac{1}{2} \|y - H\Phi \alpha\|_2^2 + \lambda \Psi(\alpha)$$

- $(P_{\text{eq}})$ is $(P_{\sigma})$ when no noise.

- $\Psi(\alpha) = \sum_{\gamma \in \Gamma} \psi(\alpha_{\gamma})$.

- $\psi$ a sparsity-promoting penalty: non-negative, continuous, even-symmetric, and non-decreasing on $\mathbb{R}^+$, not necessarily smooth at point zero to produce sparse solutions.

- e.g. $\Psi(\alpha) = \|\alpha\|_1$. 
Class of problems in CS (cont’d)

(P_\sigma), (P_\tau) and (P_\lambda) can all be cast as:

\[(P1) \quad \min_{\alpha} f_1(\alpha) + f_2(\alpha)\]

\(f_1\) and \(f_2\) are proper lsc convex functions.
Class of problems in CS (cont’d)

(P_σ), (P_τ) and (P_λ) can all be cast as:

$$(P1): \min_{\alpha} f_1(\alpha) + f_2(\alpha)$$

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(P_λ)

$$\frac{1}{2} \| y - H\Phi \alpha \|_{\ell_2}^2 + \lambda \Psi(\alpha)$$

$f_1(\alpha) = \frac{1}{2} \| y - H\Phi \alpha \|_{\ell_2}^2 \ , \ f_2(\alpha) = \lambda \Psi(\alpha)$
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\[\iota_{\Omega}(\alpha) = \begin{cases} 
0 & \text{if } \alpha \in \Omega, \\
+\infty & \text{otherwise.}
\end{cases}\]
Class of problems in CS (cont’d)

(Pσ), (Pτ) and (Pλ) can all be cast as:

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(P_\tau) & \quad \Psi(\alpha) \text{ s.t. } \| y - H\Phi\alpha \|_{\ell_2} \leq \sigma \\
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\iota_{\Omega}(\alpha) = \begin{cases} 
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Characterization

Theorem 1

(i) **Existence**: \((P1)\) possesses at least one solution if \(f_1 + f_2\) is coercive, i.e.,
\[
\lim_{\|\alpha\| \to +\infty} f_1(\alpha) + f_2(\alpha) = +\infty.
\]

(ii) **Uniqueness**: \((P1)\) possesses at most one solution if \(f_1 + f_2\) is strictly convex. This occurs in particular when either \(f_1\) or \(f_2\) is strictly convex.

(iii) **Characterization**: Let \(\alpha \in \mathcal{H}\). Then the following statements are equivalent:

(a) \(\alpha\) solves \((P1)\).

(b) \(\alpha = (I + \partial(f_1 + f_2))^{-1}(\alpha)\) (proximal iteration).

- \(\partial f_i\) is the subdifferential (set-valued map), a maximal monotone operator.

- \(J_{\partial(f_1+f_2)} = (I + \partial(f_1 + f_2))^{-1}\) is the resolvent of \(\partial(f_1 + f_2)\) (firmly non-expansive operator).
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**Operator splitting schemes**

- **Idea:** replace explicit evaluation of the resolvent of \( \partial (f_1 + f_2) \) (i.e. \( J_{\partial(f_1+f_2)} \)), by a sequence of calculations involving only \( \partial f_1 \) and \( \partial f_2 \) at a time.

- An extensive literature essentially divided into three classes:

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\( J_{\partial f_i} = (I + \partial f_i)^{-1} \)

**Moreau(-Yosida) proximity operators**
Proximity operators

The notion of a proximity operator (or inf-convolution) introduced by [Moreau 1962] as a generalization of the concept of a convex projection operator.

Definition Let \( \varphi \) be lsc proper convex. For every \( x \in \mathcal{H} \), the function \( y \mapsto \varphi(y) + \|x - y\|^2 / 2 \) achieves its infimum at a unique point denoted by \( \text{prox}_\varphi x \). The operator \( \text{prox}_\varphi : \mathcal{H} \to \mathcal{H} \) thus defined is the proximity operator of \( \varphi \). \( \forall x, p \in \mathcal{H} \)

\[
p = \text{prox}_\varphi x \iff x - p \in \partial \varphi(p) \iff p = J_{\partial \varphi} .
\]

Some properties

- The prox is uniquely valued.
- The Moreau envelope \( \varphi(p) + \|x - p\|^2 / 2 \) is Fréchet-differentiable with Lipschitz continuous gradient, hence the name Moreau-Yosida regularization.
- Many useful rules in proximal calculus (scaling, translation, quadratic perturbation, etc).
Example of proximity operator
Example of proximity operator

\[ \varphi(y) = |y| \]
Example of proximity operator
Compressed sensing optimization problems
Consider the following minimization problems:

\( (P_\sigma) : \quad \min \Psi(\alpha) \quad \text{s.t.} \quad \|y - H\Phi\alpha\|_2 \leq \sigma \)

\( (P_{eq}) : \quad \min \Psi(\alpha) \quad \text{s.t.} \quad y = H\Phi\alpha \)

\( (P_\tau) : \quad \min \frac{1}{2} \|y - H\Phi\alpha\|_2^2 \quad \text{s.t.} \quad \Psi(\alpha) \leq \tau \)

\( \Psi(\alpha) = \sum_{\gamma \in \Gamma} \psi(\alpha\gamma). \)

\( \psi \) is a sparsity-promoting penalty.
Characterizing Problem \((P_\tau)\)

\[
\min \frac{1}{2} \| y - H\Phi\alpha \|_{\ell_2}^2 \quad \text{s.t.} \quad \Psi(\alpha) \leq \tau.
\]

Proposition

(i) **Existence:** \((P_\tau)\) has at least one solution.

(ii) **Uniqueness:** \((P_\tau)\) has a unique solution if \(H\Phi\) is injective. In particular, the latter occurs when \(\Phi\) is an ortho-basis and \(\ker(H) = \emptyset\).

\(f_1\) has a Lipschitz-continuous gradient \(\Rightarrow\) Forward-Backward splitting
**FB to solve Problem** \((P_\tau)\)

**Theorem 1** Let \(\{\mu_t, t \in \mathbb{N}\}\) be a sequence such that \(0 < \inf_t \mu_t \leq \sup_t \mu_t < 2/\|\mathbf{H}\Phi\|^2\), let \(\{\beta_t, t \in \mathbb{N}\}\) be a sequence in \((0, 1]\), and let \(\{a_t, t \in \mathbb{N}\}\) and \(\{b_t, t \in \mathbb{N}\}\) be sequences in \(\mathcal{H}\) such that \(\sum_t \|a_t\| < +\infty\) and \(\sum_t \|b_t\| < +\infty\). Fix \(\alpha_0\), and define the sequence of iterates:

\[
\alpha^{t+1} = \alpha^t + \beta_t \left( \text{proj}_{B_\Psi, \tau} \left( \alpha^t + \mu_t \left( \Phi^* \mathbf{H}^* \left( y - \mathbf{H}\Phi\alpha^t \right) \right) - b_t \right) + a_t - \alpha^t \right)
\]

where \(\text{proj}_{B_\Psi, \tau}\) is the projector onto the \(\Psi\)-ball of radius \(\tau\).

(i) \(\{\alpha^{(t)}, t \geq 0\}\) converges weakly to a minimizer of \((P_\lambda)\).

(ii) For \(\Psi(\alpha) = \|\alpha\|_{\ell_1}\), there exists a subsequence \(\{\alpha^{(t)}, t \geq 0\}\) that converges strongly to a minimizer of \((P_\lambda)\).

(iii) The projection operator \(\text{proj}_{B_\Psi, \tau}\) is

\[
\text{proj}_{B_\Psi, \tau}(\alpha) = \begin{cases} 
\alpha & \text{if } \Psi(\alpha) \leq \tau, \\
\text{prox}_{\kappa_\Psi}(\alpha) & \text{otherwise},
\end{cases}
\]

where \(\kappa\) (depending on \(\alpha\) and \(\tau\)) is chosen such that \(\Psi(\text{prox}_{\kappa_\Psi}(\alpha)) = \tau\).
Proximity operators of $\Psi$

**Theorem**  Let $\Psi(\alpha) = \sum_i \psi(\alpha_i)$. Suppose that $\psi$ satisfies, (i) $\psi$ is convex even-symmetric, non-negative and non-decreasing on $[0, +\infty)$, and $\psi(0) = 0$. (ii) $\psi$ is twice differentiable on $\mathbb{R} \setminus \{0\}$. (iii) $\psi$ is continuous on $\mathbb{R}$, it is not necessarily smooth at zero and admits a positive right derivative at zero $\psi'_+(0) = \lim_{h \to 0^+} \frac{\psi(h)}{h} > 0$. Then, the proximity operator $\text{prox}_{\kappa \psi}(\alpha)$ has exactly one continuous solution decoupled in each coordinate $\alpha_i$:

$$
\hat{\alpha}_i = \text{prox}_{\kappa \psi}(\alpha_i) = \begin{cases} 
0 & \text{if } |\alpha_i| \leq \kappa \psi'_+(0), \\
\alpha_i - \kappa \psi'(\hat{\alpha}_i) & \text{if } |\alpha_i| > \kappa \psi'_+(0).
\end{cases}
$$

**Conclusion**

- Thresholding/shrinkage operator: e.g. soft-thresholding for the $\ell_1$ norm.
- Computational cost: $O(L)$ for $L$ coefficients.
Characterizing Problem \((P_\sigma)\)

\[
\min \Psi(\alpha) \text{ s.t. } \|y - H\Phi\alpha\|_{\ell^2} \leq \sigma .
\]

**Proposition**

(i) *Existence:* \((P_\sigma)\) has at least one solution.

(ii) *Uniqueness:* \((P_\sigma)\) has a unique solution if \(\psi\) is strictly convex.

\(f_1\) and \(f_2\) are proper lsc convex functions \(\Rightarrow\) **Douglas/Peaceman-Rachford** splitting
**Theorem 1** Suppose that $A = H \Phi$ is a tight frame, i.e. $AA^* = cI$. Let $\mu \in (0, +\infty)$, let $\{\beta_t\}$ be a sequence in $(0, 2)$, and let $\{a_t\}$ and $\{b_t\}$ be sequences in $\mathcal{H}$ such that $\sum_t \beta_t (2 - \beta_t) = +\infty$ and $\sum_t \beta_t (\|a_t\| + \|b_t\|) < +\infty$. Fix $\alpha_0 \in B_{\ell_2, \sigma}$ and define the sequence of iterates,

$$\alpha^{t+1/2} = \alpha^t + c^{-1} A^* (P_{B_\sigma} - I) A(\alpha^t) + b_t$$

$$\alpha^{t+1} = \alpha^t + \beta_t \left( \text{prox}_{\mu \Psi} \circ \left( 2\alpha^{t+1/2} - \alpha^t \right) + a_t - \alpha^{t+1/2} \right),$$

where,

$$(P_{B_\sigma} - I)(x) = \begin{cases} 0 & \text{if } \|x - y\|_{\ell_2} \leq \sigma, \\ x \frac{\sigma}{\|x - y\|_{\ell_2}} + y \left( 1 - \frac{\sigma}{\|x - y\|_{\ell_2}} \right) & \text{otherwise}. \end{cases}$$

Then $\{\alpha^t, t \geq 0\}$ converges weakly to some point $\alpha$ and $\alpha + c^{-1} A^* (P_{B_\sigma} - I)(\alpha)$ is a solution to $(P_\sigma)$. 
Characterizing Problem \((P_{eq})\)

\[
\min \Psi(\alpha) \quad \text{s.t.} \quad y = H\Phi\alpha.
\]

**Proposition**

(i) *Existence*: \((P_{eq})\) has at least one solution.

(ii) *Uniqueness*: \((P_{eq})\) has a unique solution if \(\psi\) is strictly convex.

(iii) *If \(\alpha\) solves \((P_{eq})\), then it solves \((P_{\sigma=0})\).*

\(f_1\) and \(f_2\) are proper lsc convex functions \(\Rightarrow\) **Douglas/Peaceman-Rachford splitting**
Theorem \textbf{Let} \( A = H\Phi \). \textbf{Let} \( \mu \in (0, +\infty) \), \textbf{let} \( \{\beta_t\} \) \textbf{be a sequence in} \( (0, 2) \), \textbf{and let} \( \{a_t\} \) \textbf{and} \( \{b_t\} \) \textbf{be sequences in} \( \mathcal{H} \) \textbf{such that} \( \sum_t \beta_t(2 - \beta_t) = +\infty \) \textbf{and} \( \sum_t \beta_t(\|a_t\| + \|b_t\|) < +\infty \). \textbf{Fix} \( \alpha_0 \) \textbf{and define} \textbf{the sequence of iterates,}

\[
\alpha^{t+1/2} = \alpha^t + A^* (AA^*)^{-1} (y - A\alpha^t) + b_t
\]

\[
\alpha^{t+1} = \alpha^t + \beta_t \left( \text{prox}_{\mu\Psi} \circ \left( 2\alpha^{t+1/2} - \alpha^t \right) + a_t - \alpha^{t+1/2} \right). \quad \text{(As before)}
\]

\textbf{Then} \( \{\alpha^t, t \geq 0\} \) \textbf{converges weakly to some point} \( \alpha \) \textbf{and} \( \alpha + A^* (AA^*)^{-1} (y - A\alpha) \) \textbf{is a solution to} \((P_{eq})\).
Other problems

The same framework can also handle problems with analysis prior $p \geq 1$:

$$\min_x \|Dx\|_p^p \quad \text{s.t.} \quad \|y - Hx\|_2 \leq \sigma$$

$$\min_x \|Dx\|_p^p \quad \text{s.t.} \quad y = Hx.$$ 

$D$ a bounded linear operator, e.g. frame $D = \Phi^*$, finite differences for TV-regularization, etc.

The proximity operator of $\|Dx\|_p^p$ obtained through Legendre-Fenchel conjugacy and Forward-Backward splitting algorithm.
Pros and cons

- $(P_\sigma)$ and $(P_{eq})$ have the same computational complexity: $2N_{iter} \left( V_H + V_\Phi + O(L) \right)$.
- $(P_\tau)$: $2N_{iter}(V_H + V_\Phi) + O(N_{iter}L \log L)$.

- $V_H$ and $V_\Phi$ are the complexities of the operators $H$ (resp. $H^*$) and $\Phi$ (resp. $\Phi^*$).
- $V_{H,\Phi}$ is $O(n)$ or $O(n \log n)$ for most useful transforms/sensing operators.

- No tight frames with $(P_{eq})$ 😞 but does not handle noise 😏.
- No tight frames with $(P_\tau)$ 😞 but $\tau$ is difficult to choose 😞.
- Tight frames with $(P_\sigma)$ 😊 but handles noise 😊 and $\sigma$ is easier to choose 😊.
Stylized applications
CS reconstruction (1)

\[ H = \text{Fourier}, \Phi = \text{DWT}, \Psi(\alpha) = \|\alpha\|_{\ell^1}, m/n = 17\% \]

Projections RealFourier m/n=0.17 SNR=30 dB

Original image

\((P_\tau)\) Iter=1000 PSNR=22.0734 dB

\((P_{cq})\)-DR Iter=1000 PSNR=23.0226 dB

\((P_{\sigma})\)-DR Iter=1000 PSNR=22.1286 dB
CS reconstruction (2)

\[ H = \text{Fourier, TV regularization, } m/n = 17\% \]

Projections Real Fourier SNR=35.2649 dB

Original phantom image

TV-DR on noiseless data Iter=200 PSNR=75.4585 dB

TVDN-DR on noisy data Iter=200 PSNR=40.7432 dB
Inpainting and CS

\[ H = \text{Dirac}, \; \Phi = \text{Curvelets + LDCT}, \; \Psi(\alpha) = \|\alpha\|_{\ell_1}, \; m/n = 20\% \]
Inpainting and CS

\(\mathbf{H} = \text{Dirac}, \ \Phi = \text{Curvelets} + \text{LDCT}, \ \Psi(\alpha) = \|\alpha\|_{\ell_1}, \ m/n = 20\%\)
Inpainting and CS

\( H = \text{Dirac}, \Phi = \text{Curvelets} + \text{LDCT}, \Psi(\alpha) = \|\alpha\|_{\ell_1}, m/n = 20\% \)
Super-resolution

Degraded image: blurred and 1:4 x 1:4 down-sampled

Original phantom image

TV-DR super-resolved image
Iter=200 PSNR=21.7344 dB
Computation time

\[ \text{CS } H = \text{Fourier, } \Phi = \text{Dirac} \]
\[ m/n = 20\%, \text{ sparsity } = 5\% \]

\[ \log_2(m) \]

\[ \text{Computation time (s)} \]

\[ \text{BP−DR, LARS, LP−Interior Point, StOMP} \]
Take-away messages

- A general framework of optimization problems in CS: maximal monotone operator splitting.

- Good and fast solvers for large-scale problems:
  - Iterative-thresholding.
  - Grounded theoretical results.

- A wide variety of applications beyond CS.
Ongoing and future work

- Beyond the linear case with AWGN [FXD-F.-JLS 07-08].
- More comparison to other algorithms.
- Work harder for algorithms faster than linear.
- Other problems, e.g. Dantzig selector.
- Other applications.
Extended experiments, code and paper available soon at 
http://www.greyc.ensicaen.fr/~jfadili

Sparsity stuff
http://www.morphologicaldiversity.org

Thanks
Any questions?